Definitions of ψ-Functions Available in Robustbase

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Preamble

Unless otherwise stated, the following definitions of functions are given by Maronna et al. (2006). Their definitions differ sometimes slightly from the ones stated here. We prefer to use another way of standardizing the functions. To avoid confusion, we first define ψ- and ρ-functions.

A *ψ-function* is defined as a function ψ such that

- ψ is odd,
- $\psi(x) > 0$ for $0 < x < \sup\{\tilde{x} : \psi(\tilde{x}) > 0\}$.
- It's derivative ψ' is 1 at 0, i.e., $\psi'(0) = 1$.

A *ρ-function* is defined as the following definite integral of a ψ-function

$$\rho(x) = \int_0^x \psi(x)dx\ .$$

A ψ-function is called *redescending* if $\psi(x) = 0$ for some $x > 0$. Corresponding to a redescending ψ-function, we define a function $\tilde{\rho}$, that is standardized such that the maximum value it attains is one. In formulas $\tilde{\rho}(x) = \rho(x)/\rho(\infty)$. $\tilde{\rho}$ is a ρ-function as defined in Maronna et al. (2006).

1 Bisquare

The bisquare family of functions is defined as,

$$\tilde{\rho}(x) = \begin{cases} 1 - [1 - (x/k)^2]^3 & \text{if } |x| \leq k \\ 1 & \text{if } |x| > k \end{cases},$$

with derivative $\tilde{\rho}'(x) = 6\psi(x)/k^2$ where,

$$\psi(x) = x \left[1 - \left(\frac{x}{k} \right)^2 \right]^2 I_{\{|x| \leq k\}} .$$

The constant for 95% efficiency of the regression estimator is 4.685 and the constant for a breakdown point of 0.5 of the S-estimator is 1.548.

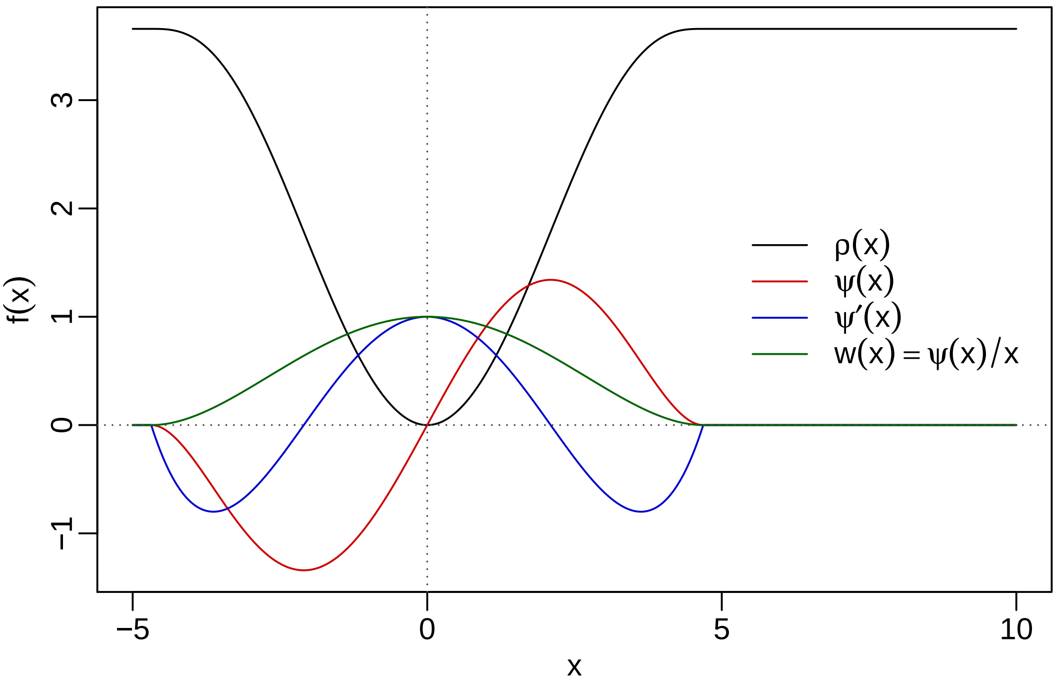

Figure 1: Bisquare family functions using tuning parameter $k = 4.685$.

2 Hampel

Let $C = \frac{1}{2}a^2 + a(b-a) + \frac{1}{2}a(c-b)$. The Hampel family of functions (Hampel et al., 1986) is defined as,

$$\tilde{\rho}(x) = \begin{cases} \frac{1}{2}x^2/C & |x| \le a \\ \left(\frac{1}{2}a^2 + (|x|-a)a\right)/C & a < |x| \le b \\ \left(\frac{1}{2}a^2 + a(b-a) + \frac{1}{2}a(|x|-b)\left(1+\frac{c-|x|}{c-b}\right)\right)/C & b < |x| \le c \\ 1 & c < |x| \end{cases},$$

$$\psi(x) = \begin{cases} x & |x| \le a \\ \mathrm{sgn}(x)a & a < |x| \le b \\ a\frac{c-|x|}{c-b}\mathrm{sgn}(x) & b < |x| \le c \\ 0 & c < |x| \end{cases}.$$

The Hampel ψ function is chosen to have a slope of 1 in the center and a slope of 1/2 in the redescending part. That implies the following relation between the three parameters a, b and c,

$$c = 2a + b\,.$$

We use $a = 1.5$, $b = 3.5$ and $c = 8$. These three parameters are then multiplied by k to get the desired efficiency / breakdown point.

The constant for 95% efficiency of the regression estimator is 0.901 and the constant for a breakdown point of 0.5 of the S-estimator is 0.212.

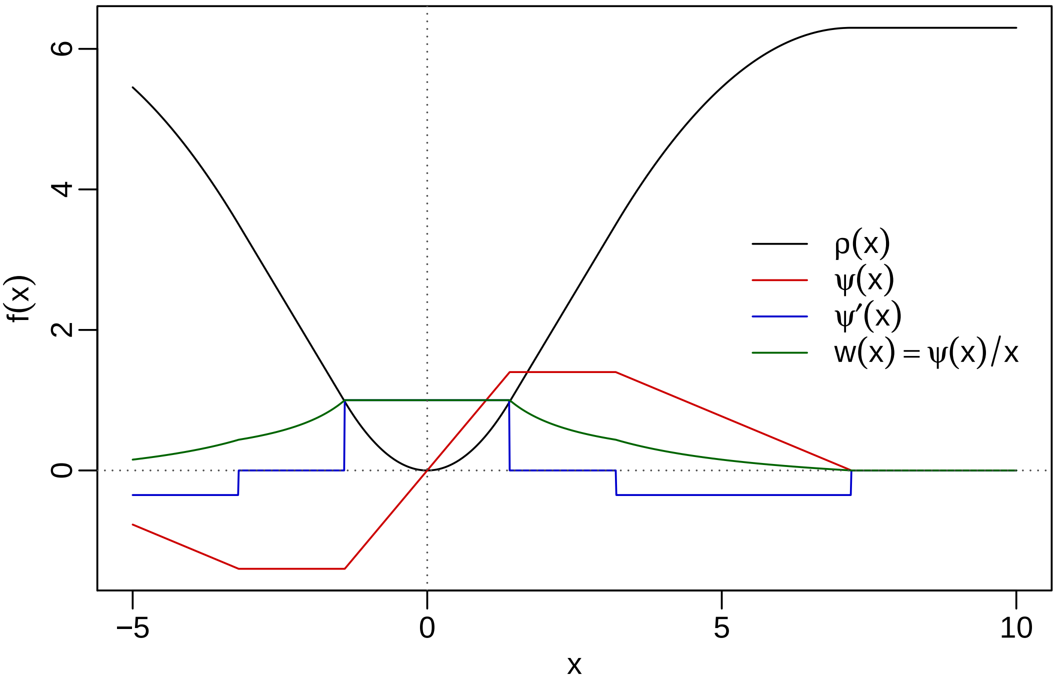

Figure 2: Hampel family of functions using tuning parameters $0.902 \cdot (1.5, 3.5, 9)$.

3 Huber

The family of Huber functions is defined as,

$$\rho(x) = \begin{cases} x^2 & \text{if } |x| \le k \\ 2k|x| - k^2 & \text{if } |x| > k \end{cases},$$

$$\psi(x) = \begin{cases} x & \text{if } |x| \le k \\ -k & \text{if } x < -k \\ k & \text{if } x > k \end{cases}.$$

The constant for 95% efficiency of the regression estimator is 1.345.

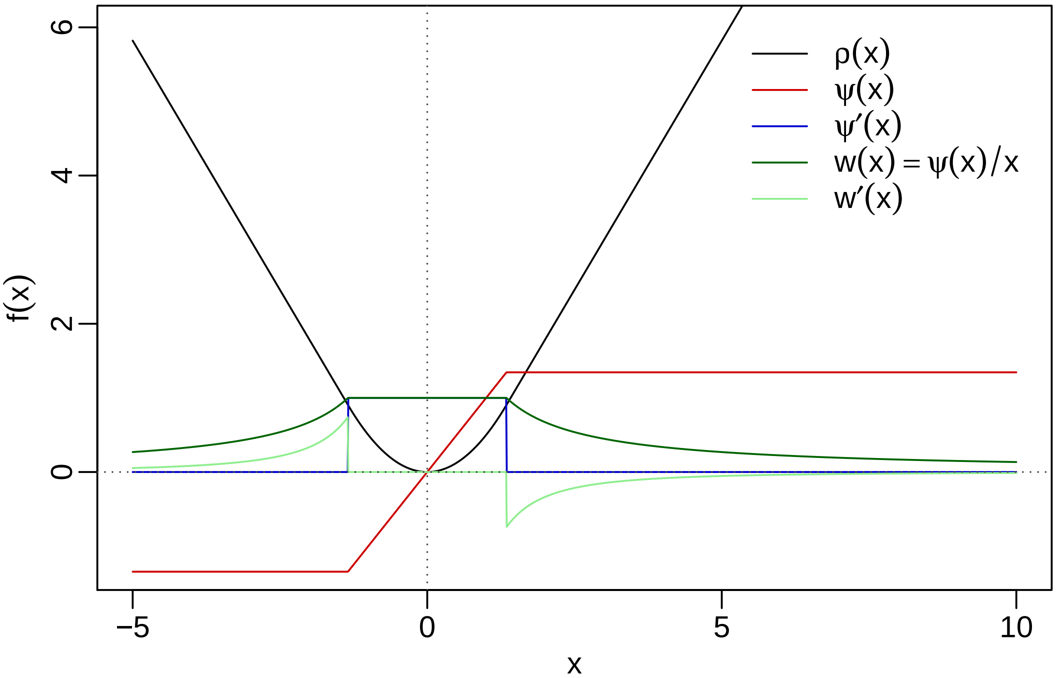

Figure 3: Huber family of functions using tuning parameter $k = 1.345$.

4 GGW

The Generalized Gauss-Weight function, or *ggw* for short, is a generalization of the Welsh ψ-function. In Koller and Stahel (2011) it is defined as,

$$\psi(x, a, b, c) = \begin{cases} x & |x| \leq c \\ \exp\left(-\frac{1}{2}\frac{(|x|-c)^b}{a}\right) x & |x| > c, \end{cases} .$$

The constants for 95% efficiency of the regression estimator are $a = 1.387$, $b = 1.5$ and $c = 1.063$. The constants for a breakdown point of 0.5 of the S-estimator are $a = 0.204$, $b = 1.5$ and $c = 0.296$.

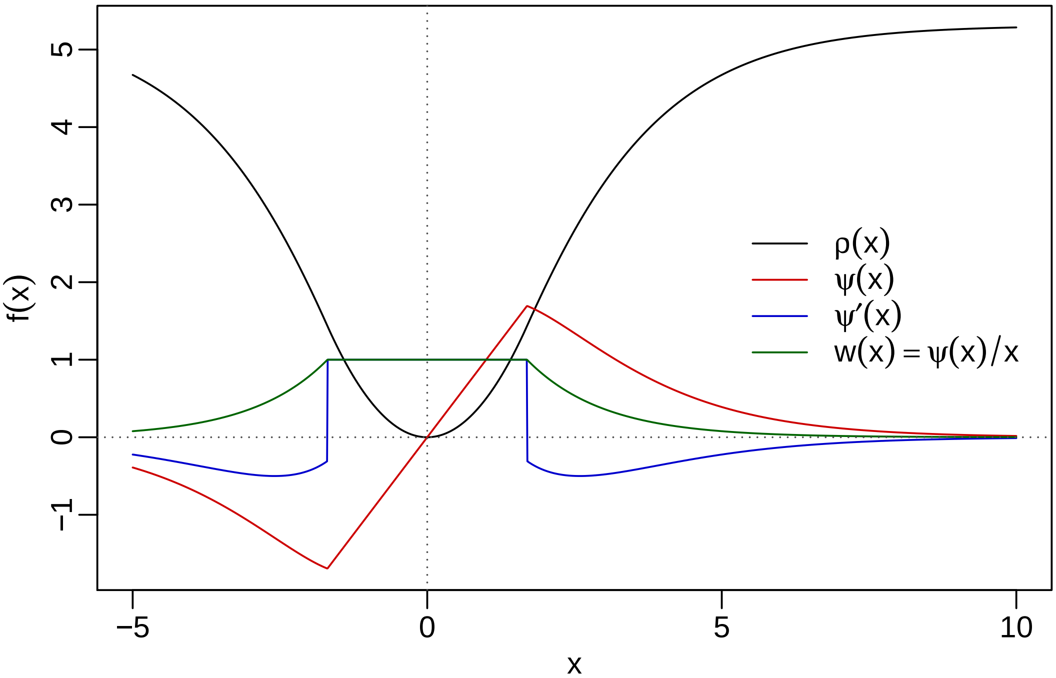

Figure 4: GGW family of functions using tuning parameters $a = 1.387$, $b = 1.5$ and $c = 1.063$.

5 LQQ

The “linear quadratic quadratic”ψ-function, or *lqq* for short, was proposed by Koller and Stahel (2011). It is defined as,

$$\psi(x) = \begin{cases} x & |x| \leq c \\ \operatorname{sign}(x)\left(|x| - \frac{s}{2b}\left(|x| - c\right)^2\right) & c < |x| \leq b + c \\ \operatorname{sign}(x)\left(c + b - \frac{bs}{2} + \frac{s-1}{a}\left(\frac{1}{2}\tilde{x}^2 - a\tilde{x}\right)\right) & b + c < |x| \leq a + b + c \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{x} = |x| - b - c$ and $a = (bs - 2b - 2c)/(1 - s)$. The parameter c determines the width of the central identity part. The sharpness of the bend is adjusted by b while the maximal rate of descent is controlled by s ($s = 1 - |\min_x \psi'(x)|$). The length a of the final descent to 0 is determined by b, c and s.

The constants for 95% efficiency of the regression estimator are $b = 1.473$, $c = 0.982$ and $s = 1.5$. The constants for a breakdown point of 0.5 of the S-estimator are $b = 0.402$, $c = 0.268$ and $s = 1.5$.

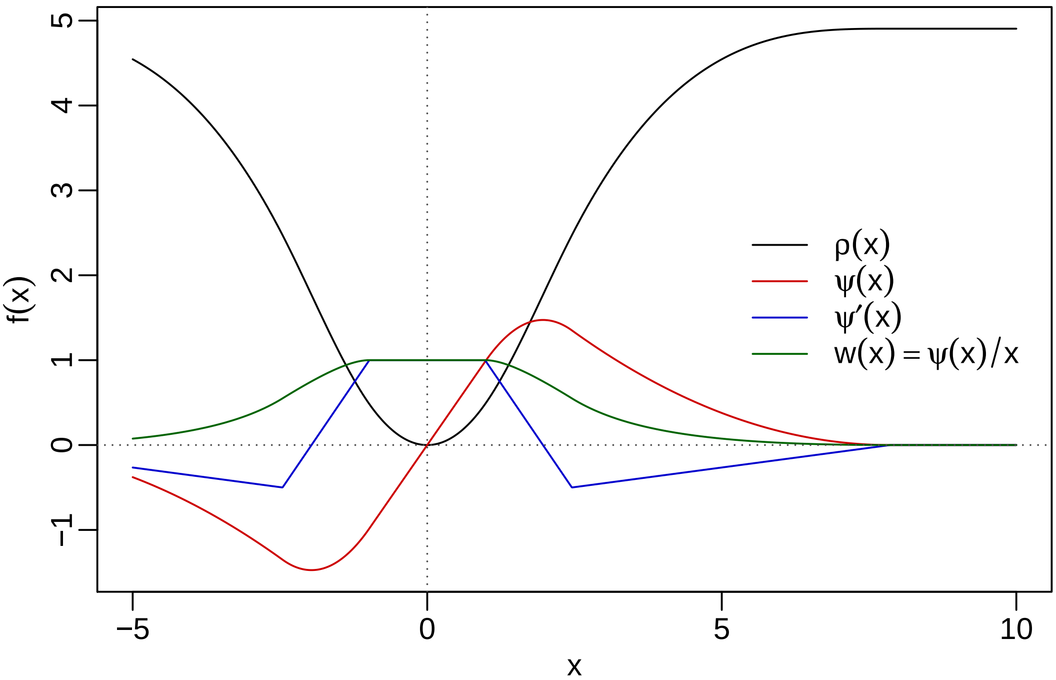

Figure 5: LQQ family of functions using tuning parameters $b = 1.473$, $c = 0.982$ and $s = 1.5$.

6 Optimal

The optimal ψ function as given by Maronna et al. (2006, Section 5.9.1),

$$\psi(x) = \operatorname{sgn}(x) \left(-\frac{\varphi'(|x|) + c}{\varphi(|x|)} \right)^{+} ,$$

where φ is the standard normal density, c is a constant and $t^{+} = \max(t, 0)$ denotes the positive part of t.

The constant for 95% efficiency of the regression estimator is 1.060 and the constant for a breakdown point of 0.5 of the S-estimator is 0.405.

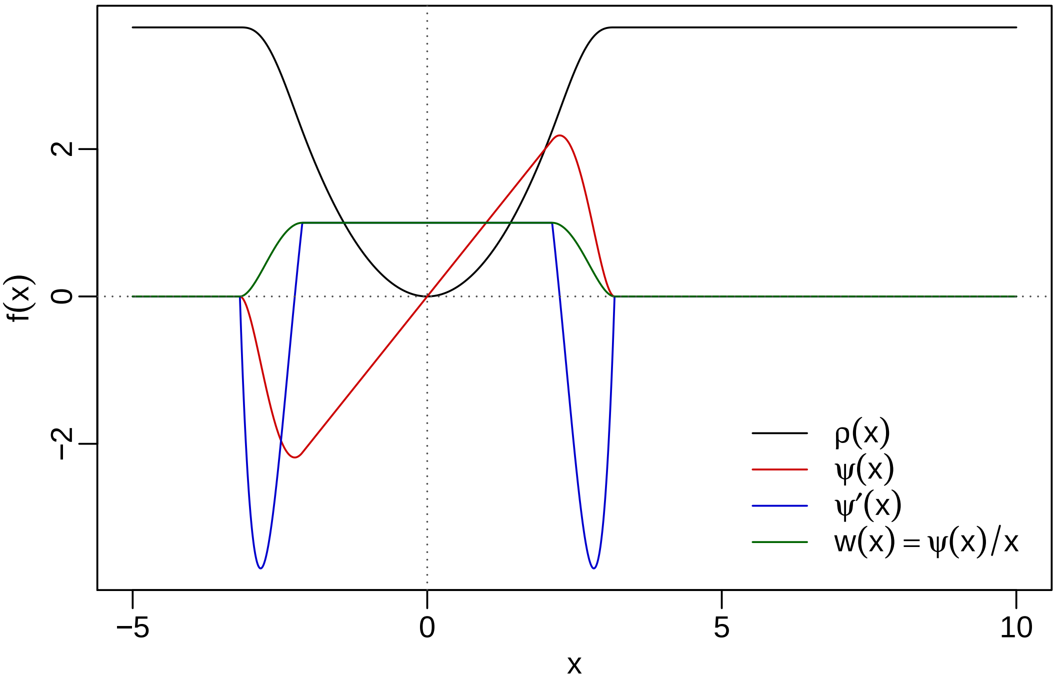

Figure 6: Optimal family of functions using tuning parameter $c = 1.06$.

7 Welsh

The Welsh ψ function is defined as (REFERENCE MISSING),

$$\tilde{\rho}(x,k) = 1 - \exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$
$$\psi(x,k) = k^2\tilde{\rho}'(x,k) = x\exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$
$$\psi'(x,k) = \left(1-\left(\frac{x}{k}\right)^2\right)\exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$

The constant for 95% efficiency of the regression estimator is 2.11 and the constant for a breakdown point of 0.5 of the S-estimator is 0.577.

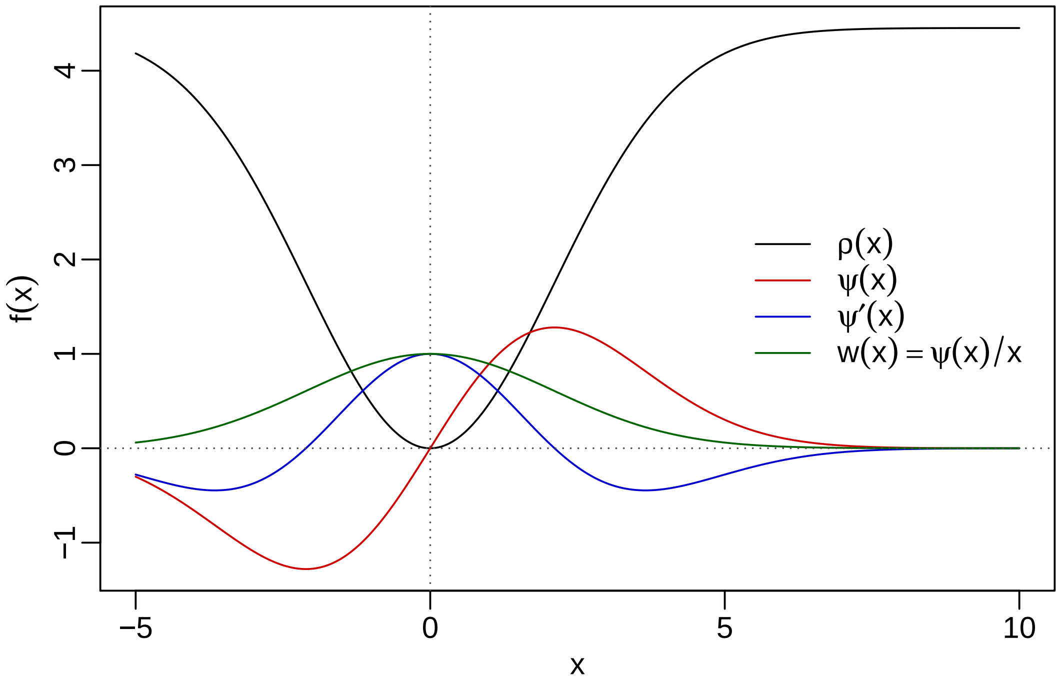

Figure 7: Welsh family of functions using tuning parameter $k = 2.11$.

References

Hampel, F., E. Ronchetti, P. Rousseeuw, and W. Stahel (1986). *Robust Statistics: The Approach Based on Influence Functions*. N.Y.: Wiley.

Koller, M. and W. A. Stahel (2011). Sharpening wald-type inference in robust regression for small samples. *Computational Statistics & Data Analysis 55*(8), 2504–2515.

Maronna, R. A., R. D. Martin, and V. J. Yohai (2006). *Robust Statistics, Theory and Methods*. N.Y.: Wiley.