

spmoran: An R package for Moran's eigenvector-based spatial regression analysis (version 2017/09)

Daisuke Murakami

Department Statistical Modeling, Institute of Statistical Mathematics

10-3 Midori-cho, Tachikawa, Tokyo, Japan

E-mail: dmuraka@ism.ac.jp

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Update: 2017/09

- A bug in the `resf_vc` function is fixed

Update: 2017/06

- Functions are modified for large data analysis (in the previous version, an error occurred when the sample size is large).
- Spatially filtered unconditional quantile regression is implemented.
- A bug in the `esf` function is fixed.

1. Introduction

Eigenvector spatial filtering (ESF; e.g., Griffith, 2003), which is also known as Moran's eigenvector mapping (MEM; e.g., Dray et al., 2006), is a regression approach to estimate and infer regression coefficients in the presence of spatial dependence. Recently, ESF is extended to random effects ESF (RE-ESF; Murakami and Griffith, 2015). RE-ESF increases the estimation accuracy of regression coefficients and their standard errors with shorter computational time. RE-ESF is also extended to spatially varying coefficient (SVC) modeling (Murakami et al., 2017). The package "spmoran" provides R functions for fast estimation of ESF and RE-ESF models with/without SVCs.

This tutorial applies ESF and RE-ESF to a land price analysis of flood hazard. The target area is Ibaraki prefecture, Japan. Explained variables are logged land prices in 2015 (JPY/m²; sample size: 647; Figure 1). Explanatory variables are as listed in Table 1. All these variables are downloaded from the National Land Numerical Information download service (NLNI; <http://nlftp.mlit.go.jp/ksj-e/index.html>).

The following is a data image, in which "px" and "py" are spatial coordinates:

```
> data <- read.csv("data.csv")
```

```
> data[ 1:6, ]
```

	px	py	ln_price	station	tokyo	city	flood
1	19235.25	-4784.562	10.126631	4.0109290	43.38504	1	1.5
2	16450.37	-8782.851	10.835652	0.8977986	43.38504	1	0.0
3	17673.30	-8351.802	10.633449	0.5596742	43.38504	1	0.0
4	17824.50	-7704.343	9.878170	0.8504618	43.38504	0	0.0
5	67334.31	58001.724	10.122623	3.1660661	140.95839	1	0.0
6	68929.42	55028.751	9.952278	2.5008292	140.95839	1	1.5

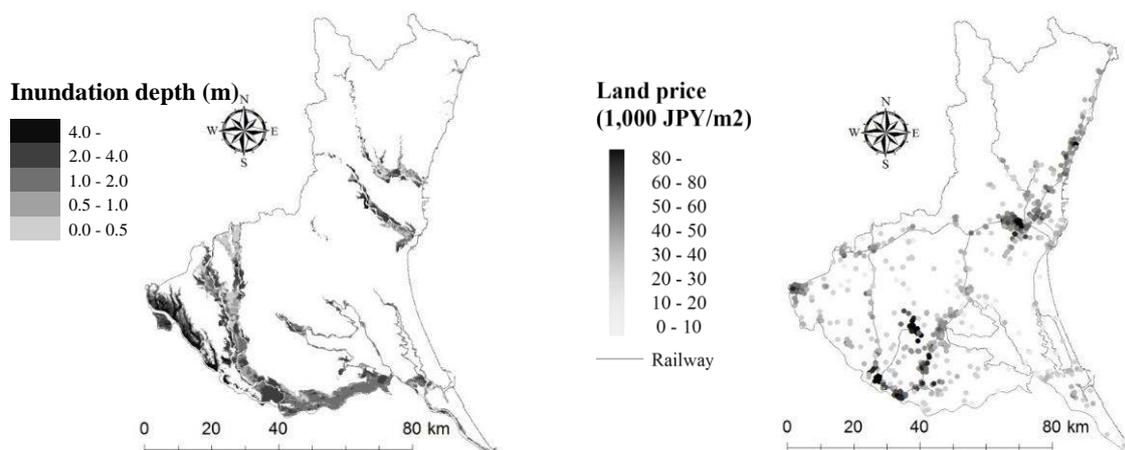


Figure 1. Anticipated inundation depth (left) and officially assessed land prices in 2015 (right) in the Ibaraki prefecture

Table 1. Explanatory variables

Variables	Description
tokyo	Logarithm of the distance from the nearest railway station to Tokyo Station [km]
station	Logarithm of the distance to the nearest railway station [km]
flood	Anticipated inundation depth [m]
city	1 if the site is in an urban promotion land and 0 otherwise

ESF/RE-ESF are implemented in the following two steps:

- Extraction of Moran's eigenvectors (see Section 2);
- Parameter estimation of the ESF/RE-ESF model (see Sections 3 and 4).

Sections 2, 3 and 4 explain implementation of these steps, while Section 5 explains how to accelerate the computation.

2. Extraction of Moran's eigenvectors

Consider a doubly-centered spatial connectivity matrix, \mathbf{MCM} , where \mathbf{C} is a symmetric spatial proximity matrix whose diagonals are zeros, $\mathbf{M} = \mathbf{I} - \mathbf{1}\mathbf{1}'/N$ is a centering operator, where \mathbf{I} is an identity matrix, and $\mathbf{1}$ is a vector of ones, and N is the sample size. The eigenvectors, $\mathbf{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_N\}$, of \mathbf{MCM} furnish all possible distinct map pattern descriptions of latent spatial dependence, with each level being indexed by the Moran coefficient (MC; Griffith, 2003; Tiefelsdorf and Griffith, 2007). Eigenvectors corresponding to large positive eigenvalue describe map patterns with greater positive spatial dependence (i.e. greater positive MC), whereas eigenvectors corresponding to negative eigenvalue describe map patterns with negative spatial dependence. As positive spatial dependence is dominant in most real-world cases, only eigenvectors with positive eigenvalues are considered in many applied studies.

The function `meigen` extracts eigenvectors corresponding to positive eigenvalue (i.e. $\lambda_l > 0$, where λ_l is the l -th eigenvalue)¹. The command is as follows:

```
> coords <- data[,c("px", "py")]
> meig <- meigen(coords = coords)
```

Calculated eigenvectors and eigenvalues are displayed by commanding `meig$sf` and `meig$ev`, respectively. By default, \mathbf{C} is given by the matrix whose (i, j) -th element equals $\exp(-d_{i,j}/r)$, where

¹ For the distance-based \mathbf{C} , it is standard to set the threshold by $\lambda_l > 0$, which attempts to consider all elements describing positive spatial dependence.

d_{ij} is the Euclidean distance between sites i and j , and r is the longest distance in the minimum spanning tree covering the sample sites (Dray et al., 2006; Murakami and Griffith, 2015).

The distance-based \mathbf{C} may be replaced with other types of spatial connectivity matrix. In this case, user must construct the matrix *a priori*. For example, the following command employs the 4-nearest-neighbor-based \mathbf{C} :

```
> library( spdep )
> col.knn <- knearneigh( coordinates( coords ), k = 4 )
> cmat <- nb2mat( knn2nb( col.knn ), style = "B" )
> meigB <- meigen( cmat = cmat )
```

If the spatial connectivity matrix is not symmetric like the 4-nearest neighbor-based \mathbf{C} , `meigen` symmetrizes it by taking $\{\mathbf{C} + t(\mathbf{C})\}/2$. In cases with binary connectivity-based \mathbf{C} (e.g. proximity-based \mathbf{C} ; k -nearest-neighbor-based \mathbf{C}), $\lambda_l / \lambda_1 > 0.25$ is a standard threshold for the eigenvector extraction². The thresholding is implemented by the following command:

```
> meigB <- meigen( cmat = cmat, threshold = 0.25 )
```

The eigen-decomposition can be very slow for large samples. To accelerate the computation, the function `meigen_f` approximates the eigenvectors by applying the Nystrom extension, which is a dimension reduction technique (Murakami and Griffith, 2017)³. The command is as follows:

```
> meig_f <- meigen_f( coords = coords )
```

Just like `meigen`, `meig_f$sf` and `meig_f$ev` return approximated eigenvectors and eigenvalues, respectively. By default, the first 200 eigenvectors are approximated⁴. While `meigen` takes 243.79 seconds for the exact eigen-decomposition, `meigen_f` takes only 0.38 seconds (see Section 5 for further details).

² The threshold $\lambda_l / \lambda_1 > 0.25$ attempts to capture roughly 5% of the variance in explained variables attributable to positive spatial dependence (Griffith and Chun, 2014).

³ This approximation is available only for the distance-based \mathbf{C} .

⁴ Consideration of 200 eigenvectors is recommended because Murakami and Griffith (2017) show that the approximation error in regression coefficients is quite small when 200 (or more) eigenvectors are considered while the error increases in cases with fewer than 200 eigenvectors.

3. ESF and RE-ESF models

3.1. ESF model

The linear ESF model is formulated as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I}),$$

where \mathbf{E} is a matrix whose l -th column is the l -th eigenvector, \mathbf{e}_l , and $\boldsymbol{\gamma}$ is a vector of coefficients. This model is identical to the standard linear regression model.

The ESF model is estimated using the following steps: (i) eigenvectors whose eigenvalue exceeds a threshold are extracted from **MCM**; (ii) stepwise eigenvector selection is performed; (iii) the ESF model with selected eigenvectors is estimated by ordinary least squares.

The following command estimates the linear ESF model. In step (i), following many ESF studies (Griffith, 2003; Tiefelsdorf and Griffith, 2007), eigenvectors whose eigenvalue fulfills $\lambda_l/\lambda_1 > 0.25$ are extracted from a binary connectivity-based **C** (4-nearest-neighbor-based **C**; see Section 2):

```
> y      <- data[ , "ln_price" ]                # Explained variables
> x      <- data[ , c( "station", "tokyo", "city", "flood " ) ] # Explanatory variables
> meig   <- meigB                              # Moran's eigenvectors (knn-based C)
> e_res  <- esf( y = y, x = x, meig = meig, vif = 10, fn = "r2" )
```

To cope with possible multicollinearity, eigenvectors are selected so that the variance inflation factor (VIF), which is an indicator of multicollinearity, does not exceed 10. It is implemented by setting `vif = 10`, whereas VIF is not considered by default. The eigenvector selection is performed by the adjusted R^2 maximization (`fn = "r2"`; default). Akaike information criterion (AIC) minimization (`fn = "aic"`) or Bayesian Information criterion (BIC) minimization (`fn = "bic"`). Alternatively, all eigenvectors are considered without selecting them by setting `fn = "all"`.

When `fn = "r2"`, the coefficient estimates yield:

```
> e_res$b
              Estimate          SE      t_value      p_value
(Intercept) 9.932080e+00 0.0587240255 169.13146372 0.000000e+00
station     -6.911515e-02 0.0065601988 -10.53552610 5.070594e-24
tokyo       -2.846888e-05 0.0004214075  -0.06755664 9.461599e-01
city         6.738630e-01 0.0360500253  18.69244166 2.121536e-62
flood       2.795299e-02 0.0142681894   1.95911280 5.053884e-02
```

Station (-) and city (+) are statistically significant at the 0.1% level. It is verified that urban areas

with good access to a railway station have a higher land price than other areas. We can see that flood is positively significant at the 10% level. This result suggests that influence from flood disaster, which is expected to be negative, is not appropriately reflected to land price.

VIF values are displayed by the following command:

```
> e_res$vif
              VIF
station  1.367917
tokyo    1.225594
city     1.282930
flood    1.208189
sf4      1.167728
sf9      1.017697
sf12     1.142611
sf31     1.084662
sf33     1.032077
sf45     1.035118
sf32     1.095973
sf26     1.012234
sf6      1.059948
sf20     1.016059
```

The following command displays error statistics, including residual standard error (residual_SE), adjusted R^2 (adjR2), log-likelihood (logLik), AIC, BIC, and degrees of freedom (DF):

```
> e_res$e
              stat
resid_SE  0.3542671
adjR2     0.6987400
logLik   -239.0702859
AIC      510.1405718
BIC      581.6981125
```

While we have discussed ESF with binary connectivity-based C , which is popular in regional science, ESF with distance-based C , which is popular in ecology, is implemented as follows:

```
> meig <- meigen( coords=coords ) #Moran's eigenvectors (distance-based C)
> e_res <- esf( y=y, x=x, meig=meig, fn = "r2" )
```

The distance-based ESF is often referred to as MEM or a principal coordinate neighborhood matrix (PCNM) (see Legendre and Legendre, 2012).

A major disadvantage of ESF is the computational cost. To cope with this problem, Murakami and Griffith (2017) develops a fact approximation. It is implemented by the following command:

```
> meig_f <- meigen_f( coords = coords )
> e_res <- esf( y = y, x = x, meig = meig_f, fn = "all" )
```

Here, all eigenvectors in `meig_f` are considered without selecting them by setting `fn = "all"`. It is sufficient for thousands or more samples (Murakami and Griffith, 2017).

3.2.RE-ESF model

The RE-ESF model is formulated as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{E}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\gamma} \sim N(\mathbf{0}, \sigma_\gamma^2 \boldsymbol{\Lambda}(\alpha)), \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}).$$

Unlike ESF, $\boldsymbol{\gamma}$ is given by a vector of random coefficients: $\boldsymbol{\gamma} \sim N(\mathbf{0}, \sigma_\gamma^2 \boldsymbol{\Lambda}(\alpha))$. $\boldsymbol{\Lambda}(\alpha)$ is a diagonal matrix whose elements are the eigenvalues, which are multiplied by α . σ_γ^2 and α represent the variance and the scale of the spatially dependent component; large α implies global-scale spatial variation, while small α implies local variation. These parameters act as shrinkage parameters controlling variance inflation.

The RE-ESF model is estimated using the following steps: (i) eigenvectors whose eigenvalue exceeds a threshold are extracted from **MCM**; (ii) parameters are estimated by the maximum likelihood (ML) method or the restricted maximum likelihood (REML) method. REML estimation is preferable because it accounts for the degrees of freedom lost by estimating the regression coefficients.

The REML estimation is implemented by the following command:

```
> meig <- meigen( coords = coords ) #Moran's eigenvectors (distance-based C)
> r_res <- resf( y = y, x = x, meig = meig, method = "reml" )
```

ML is implemented by replacing `method = "reml"` with `method = "ml"`.

Estimated coefficients are displayed as follows:

```
> r_res$b
```

	Estimate	SE	t_value	p_value
(Intercept)	9.9902998898	0.169833051	58.8242385	0.000000e+00
station	-0.0792859163	0.009598674	-8.2600901	8.881784e-16
tokyo	-0.0003715008	0.001795810	-0.2068709	8.361807e-01
city	0.6857752216	0.036926493	18.5713608	0.000000e+00
flood	-0.0043670379	0.014784271	-0.2953841	7.678025e-01

Just like the estimates for ESF, station (-) and city (+) are statistically significant, and tokyo is not. In contrast, unlike ESF, flood is not statistically significant. Because RE-ESF tends to outperform ESF in terms of the estimation accuracy of regression coefficients and their standard errors (Murakami and Griffith, 2015), the results of RE-ESF might be more reliable. Error statistics are extracted by the following command:

```
> r_res$e
```

	stat
resid_SE	0.3116825
adjR2(cond)	0.7649824
rlogLik	-262.9627231
AIC	543.9254462
BIC	584.1765628

where `adjR2(cond)` is the adjusted conditional R^2 , and `rlogLik` is the restricted log-likelihood. `rlogLik` is replaced with `loglik`, which denotes log-likelihood, if `method = "ml"`. It is important to note that, when REML is used, AIC and BIC are comparable only with models with the same explanatory variables. `resf` also returns the estimated shrinkage parameters as follows:

```
> r_res$s
```

	par
shrink_sf_SE	0.4337118
shrink_sf_alpha	0.2449076

where `shrink_sf_SE` and `shrink_sf_alpha` are σ_γ and α , respectively. The standard error of the

spatially dependent component ($\text{shrink_sf_SE} = 0.4337118$)⁵ is greater than the residual standard error ($\text{resid_SE} = 0.3116825$). In other words, substantial spatial dependent variations, which are ignored if the linear regression model is estimated, are captured by $\mathbf{E}\boldsymbol{\gamma}$. shrink_sf_alpha is smaller than one. This implies that coefficients on each eigenvector are shrunk comparatively equally, irrespective of their corresponding eigenvalues. The resulting $\mathbf{E}\boldsymbol{\gamma}$ has local-scale spatial variations relative to $\mathbf{E}\boldsymbol{\gamma}$ with large shrink_sf_alpha .

`resf` performs the computationally efficient ML/REML estimation of Murakami and Griffith (2017). The command is as follows:

```
> meig_f <- meigen_f( coords = coords )
> r_res <- resf( y = y, x = x, meig = meig_f, method = "reml" )
```

4. Extended models

4.1. Spatially varying coefficients (SVC) model

Murakami et al. (2017) suggest that a RE-ESF-based SVC modeling outperforms geographically weighted regression (GWR), which is the standard approach for SVC modeling, in terms of coefficient estimation accuracy and computational time.

The RE-ESF-based SVC model is formulated as follows:

$$\mathbf{y} = \sum_k \mathbf{x}_k \otimes \boldsymbol{\beta}_k + \mathbf{E}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\beta}_k = \beta_{k,0}\mathbf{1} + \mathbf{E}\boldsymbol{\gamma}_k, \quad \boldsymbol{\gamma}_k \sim N(\mathbf{0}, \sigma_{\gamma,k}^2 \boldsymbol{\Lambda}(\alpha_k)), \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma^2 \mathbf{I}),$$

where $\boldsymbol{\beta}_k$ is the vector of SVCs on the k th explanatory variables, \mathbf{x}_k . $\boldsymbol{\beta}_k$ consists of the constant component, $\beta_{k,0}\mathbf{1}$, and the spatially varying component, $\mathbf{E}\boldsymbol{\gamma}_k$. The latter is modeled by Moran's eigenvectors, \mathbf{E} , and their random coefficients, $\boldsymbol{\gamma}_k \sim N(\mathbf{0}, \sigma_{\gamma,k}^2 \boldsymbol{\Lambda}(\alpha_k))$. $\boldsymbol{\Lambda}(\alpha_k)$ is a diagonal matrix whose elements are the eigenvalues, which are multiplied by α_k . $\sigma_{\gamma,k}^2$ denotes the variance of the spatially dependent component, $\mathbf{E}\boldsymbol{\gamma}_k$, whereas α_k denotes the spatial scale of the component; large/small α_k implies global/local-scale spatial variation explained by $\mathbf{E}\boldsymbol{\gamma}_k$. These parameters act as shrinkage parameters controlling variance inflation. An interesting point is that, unlike GWR, the RE-ESF-based approach estimates the spatial scale of each SVC using α_k .

⁵ The following relationship holds: $\text{Var}[\mathbf{E}\boldsymbol{\gamma}] = \mathbf{E}\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{E}' = \sigma_{\gamma}^2 \mathbf{E}\boldsymbol{\Lambda}(\alpha)\mathbf{E}' = \sigma_{\gamma}^2 \hat{\mathbf{C}}_M^{\alpha}$, where $\hat{\mathbf{C}}_M^{\alpha}$ is $\mathbf{M}\mathbf{C}^{\alpha}\mathbf{M}$ approximated by the eigenvectors in \mathbf{E} . Hence, σ_{γ}^2 denotes the variance of the spatially dependent component.

In this tutorial, coefficients on station, city, and flood are allowed to vary across geographical space whereas coefficients on Tokyo are not. The command for the SVC modeling is as

```
> xv    <- x[,c( "station", "city", "flood" )]      #x with spatially varying coefficients
> xconst <- x[, "tokyo" ]                          #x with constant coefficients
> meig   <- meigen( coords = coords )              #Moran's eigenvectors (distance-based C)
> rv_res <- resf_vc( y = y, x = xv, xconst = xconst, meig = meig, method = "reml" )
```

The constant coefficient estimate for tokyo is returned by the following command:

```
> rv_res$b
      Estimate      SE      t_value      p_value
V1 -0.0009924332  0.001782719 -0.5566962  0.5779578
```

As with the output from the basic RE-ESF model, tokyo is statistically insignificant. Considering computational cost and stability, it might be preferable to employ SVCs on at most around four explanatory variables, and constant coefficients on the other explanatory variables (see Section 5).

Estimated SVCs and their p -values are displayed by the following command:

```
> rv_res$b_vc[ 1:6, ]
      (Intercept)      station      city      flood
1  9.875385 -0.06311678  0.5690735  0.006637360
2  10.278009 -0.11503321  0.8255947  0.005446833
3  10.173544 -0.10025270  0.7743310  0.006120595
4  10.138267 -0.09395701  0.7445319  0.006262945
5  10.207279 -0.10122246  0.5212322 -0.058020901
6  10.258219 -0.08688370  0.5006614 -0.059386765

> rv_res$p_vc[ 1:6, ]
      (Intercept)      station      city      flood
1      0  0.288107321  1.324063e-06  0.76500129
2      0  0.006000605  6.344747e-11  0.79813709
3      0  0.012536479  6.735013e-10  0.77200809
4      0  0.019306686  3.431317e-10  0.76597759
5      0  0.058577563  5.667240e-05  0.04008828
6      0  0.124107522  1.405513e-03  0.15243568
```

They can be summarized as follows:

```
> summary( rv_res$b_vc )
```

	(Intercept)	station	city	flood
Min. :	8.909	Min. : -0.21020	Min. : -0.02115	Min. : -0.066797
1st Qu.:	9.831	1st Qu.: -0.15448	1st Qu.: 0.57226	1st Qu.: -0.049578
Median :	10.062	Median : -0.12184	Median : 0.68319	Median : -0.013046
Mean :	10.061	Mean : -0.11572	Mean : 0.67039	Mean : -0.021668
3rd Qu.:	10.242	3rd Qu.: -0.07764	3rd Qu.: 0.81286	3rd Qu.: 0.003591
Max. :	10.946	Max. : 0.06522	Max. : 1.06872	Max. : 0.009442

```
> summary( rv_res$p_vc )
```

	(Intercept)	station	city	flood
Min. :	0	Min. : 0.000001	Min. : 0.0000000	Min. : 0.003934
1st Qu.:	0	1st Qu.: 0.001426	1st Qu.: 0.0000001	1st Qu.: 0.086623
Median :	0	Median : 0.010549	Median : 0.0000068	Median : 0.585556
Mean :	0	Mean : 0.123792	Mean : 0.0171369	Mean : 0.495177
3rd Qu.:	0	3rd Qu.: 0.175201	3rd Qu.: 0.0006345	3rd Qu.: 0.853780
Max. :	0	Max. : 0.945239	Max. : 0.9582583	Max. : 0.995845

The result suggests that the spatially varying intercept and SVCs on city are positively significant across the target area. station is negatively significant in many sample sites, and flood is statistically insignificant in most sample sites.

Figure 2 displays the estimated coefficients and their statistical significance. Estimated SVCs on station demonstrate that the distance to a railway station has a significant influence on land price in areas along railways. SVCs on city are positively significant across the target area. SVCs on flood suggest that flood risk is negatively significant around Mito city, which is the prefectural capital. Mito city has a long history as a castle town. The negative sign on flood might be because Mito city has adapted to flood disaster in its long history.

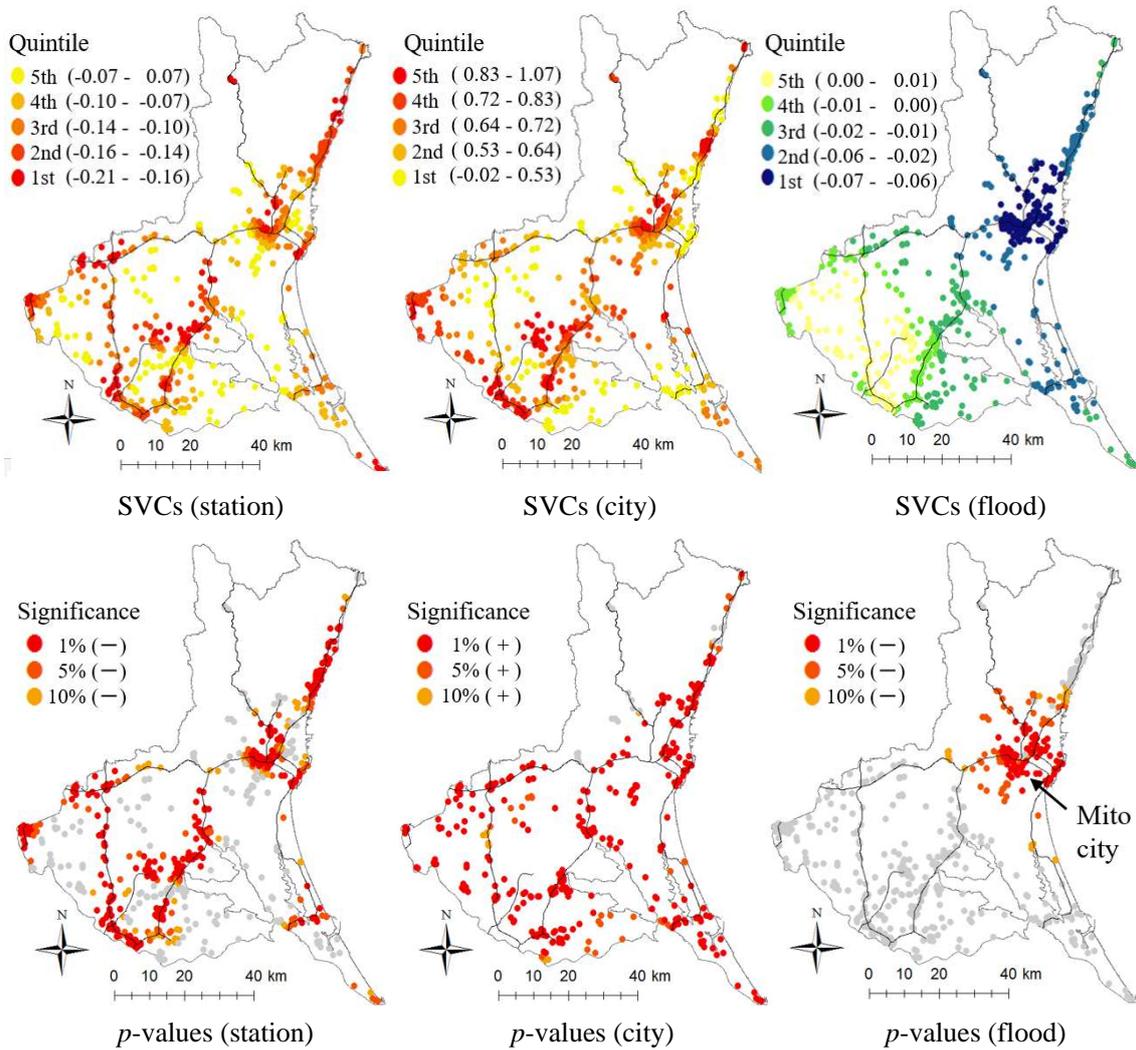


Figure 2. Estimated SVCs and their p -values (the spatially varying intercept is omitted)

Just like `resf`, `resf_vc` returns shrinkage parameter estimates for SVCs. In our case, the estimates are as follows:

```
> rv_res$s
```

	(Intercept)	station	city	flood
Shrink_sf_SE	0.4562311	0.0820578615	0.30293756	0.04153263
Shrink_sf_alpha	0.1472938	0.0001043149	0.04450748	1.59444026

`Shrink_sf_SE` summarizes the estimated standard errors, $\sigma_{\gamma,k}$, for each SVC, and `Shrink_sf_alpha` summarizes the estimated α_k parameters. Large α_k values imply strong shrinkage for local variations. For example, SVCs on flood have a global map pattern due to the large α_k value while SVCs on station have a local pattern due to the small α_k value. Thus, the α_k parameter controls the spatial scale

of the k -th SVCs.

Error statistics for the SVC model are displayed by the following command although $\log\text{Lik}$ and AIC are reference values because we apply the REML estimation:

```
> r_res$e
              stat
resid_SE      0.2637017
adjR2(cond)   0.8312410
rlogLik       -230.4469132
AIC           482.8938264
BIC           532.0896357
```

4.2. Spatially filtered unconditional quantile regression (SF-UQR)

While the conventional conditional quantile regression (CQR) estimates the influence of x_k on the τ -th “conditional” quantile of y , $q_\tau(y|x_k)$, the unconditional quantile regression (UQR; Firpo et al., 2009) estimates the influence of x_k on the “unconditional” quantile of y , $q_\tau(y)$.

Suppose y and x_k represent land price and accessibility, respectively. UQR estimates the influence of accessibility on land price in each price range. This interpretation does not hold for CQR, because it quantifies the influence of accessibility on land prices conditional on x_k (see, **Figure 3**). Thus, UQR coefficients are more interpretable than CQR coefficients.

In this context, Murakami and Seya (2017) developed the spatial filter UQR (SF-UQR). The SF-UQR model is formulated as follows:

$$\mathbf{r}_\tau = \mathbf{X}\boldsymbol{\beta}_\tau + \mathbf{E}\boldsymbol{\gamma}_\tau + \boldsymbol{\varepsilon}, \quad \boldsymbol{\gamma}_\tau \sim N(\mathbf{0}, \sigma_{\boldsymbol{\gamma}_\tau}^2 \boldsymbol{\Lambda}(a_\tau)), \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma_\tau^2 \mathbf{I}),$$

where \mathbf{r}_τ is a vector whose i -th element equals the re-centered influence function (RIF) for the i -th explained variable, y_i . The SF-UQR is a UQR considering spatial dependence.

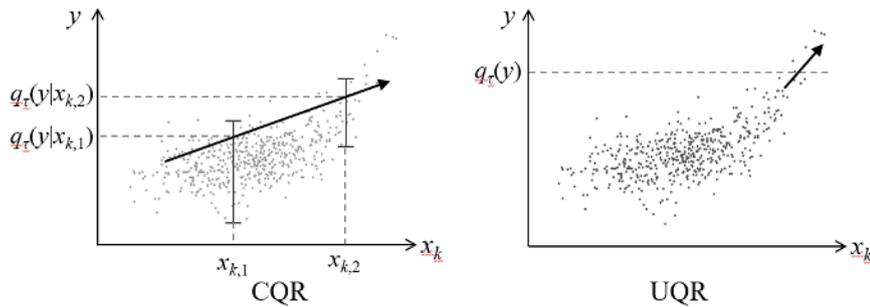


Figure 3: CQR and UQR coefficients. Allows illustrate their coefficients on the 0.9 quantile. CQR coefficient equals $\partial q_\tau(y|x_k)/\partial x_k$ whereas UQR coefficient equals $\partial q_\tau(y)/\partial x_k$.

The `spmoran` package provides the `resf_qr` function to estimate the SF-UQR model. The command is as follows:

```
> qr_res <- resf_qr( y = y, x = x, meig = meig, boot = T )
```

If `boot = T`, a semiparametric bootstrapping is performed to estimate the standard errors of UQR coefficients, and these are not calculated if `boot = F`. This function returns parameters estimated at 0.1, 0.2, ..., 0.9 quantiles by default. The quantile(s) can be specified by using an argument `tau`; for example, parameters at the 0.22 quantile are estimated by assigning `tau = 0.22`.

The computational complexity for the bootstrap iterations does not depend on the sample size, N (see, Murakmai and Griffith, 2017), but it depends on the number of eigenpairs, L , which grows as N increases. Hence, for large samples, it is useful to restrict L as follows:

```
> meig <- meig( coords, enum = 200 )  
> qr_res <- resf_qr( y = y, x = x, meig = meig, boot = T )
```

For very large N , which prohibits the eigen-decomposition, the following eigen-approximation would be useful:

```
> meig <- meig_f( coords ) #It approximates the first 200 eigen-pairs by default
```

UQR coefficients estimated by the `resf_qr` function can be visualized by the `plot_qr` function. The commands to plot estimated coefficients for the first five explanatory variables are as follows:

```
> plot_qr( qr_res, 1 )  
> plot_qr( qr_res, 2 )  
> plot_qr( qr_res, 3 )  
> plot_qr( qr_res, 4 )  
> plot_qr( qr_res, 5 )
```

The numbers 1 to 5 specify which regression coefficients are plotted (1: intercept). The resulting plots are as follows:

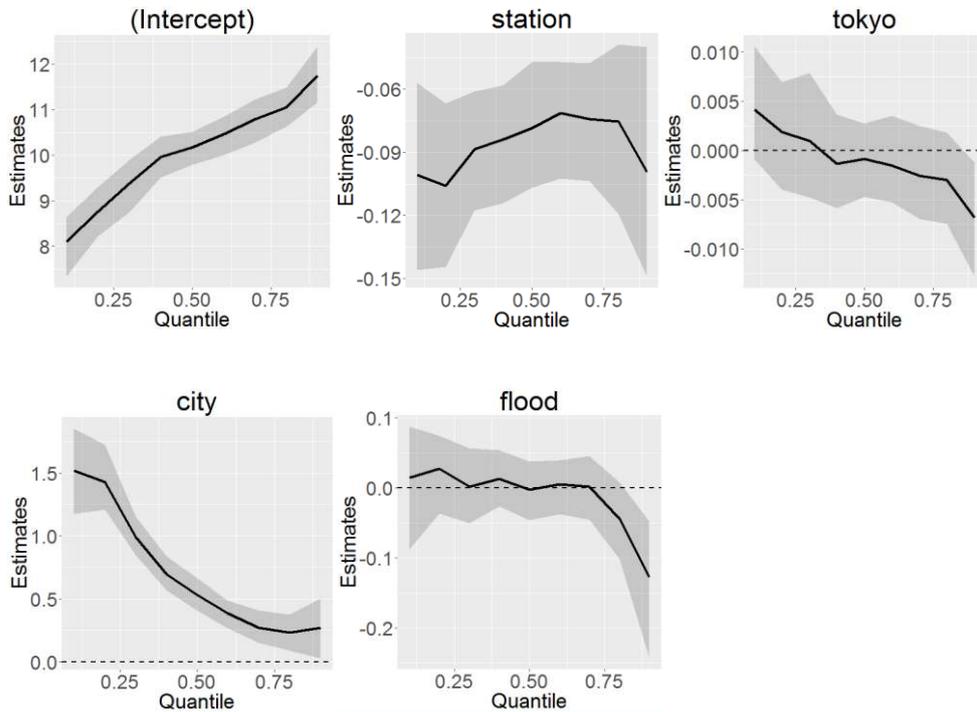


Figure 4. Outputs from the `plot_qr` function (regression coefficients). Solid lines are coefficient estimates and gray areas are their 95% confidential intervals.

On the other hand, the standard errors for the residual spatial dependent component (`shrink_sf_SE`) are plotted by assigning `pnum = 1` and `par = "s"`, while the scale (degree) parameters for the component (`shrink_sf_alpha`) are plotted by assigning `pnum = 2` and `par = "s"`. The commands and the outcomes are as follows:

```
> plot( qr_res, 1, "s" )
> plot( qr_res, 2, "s" )
```

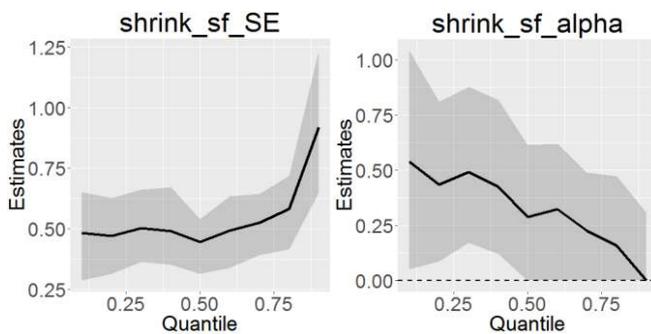


Figure 5. Outputs from the `plot_qr` function (shrinkage (variance) parameters). Solid lines are coefficient estimates and gray areas are their 95 % confidential intervals.

Parameter estimates are displayed by the following commands:

```
> res$b
> res$s
```

When `boot = T`, parameter estimates, lower and upper bounds for their 95% confidential intervals, and p-values are returned by the following command:

```
> res$B
> res$S
```

Error statistics, including the residual standard error and the adjusted quasi conditional R^2 , are displayed as follows:

```
> res$e
```

	tau=0.1	tau=0.2	tau=0.3	...	tau=0.9
resid_SE	0.93164	0.67819	0.58475	...	1.0025
quasi_adjR2(cond)	0.43749	0.57931	0.57318	...	0.4258

5. Tips for fast computation

5.1. Eigen-decomposition

As discussed, `meigen_f` performs a fast eigen-approximation, and extracts the first 200 eigenvectors by default. The computation is further accelerated by reducing number of approximated eigenvectors. It is achieved by setting `enum` by a positive integer less than 200. For example, in the case with 5000 samples and `enum = 200` (default), 100, and 50, computational times are as follows:

```
> coords_test <- cbind( rnorm( 5000 ), rnorm( 5000 ) )
```

```
-----CP time (without approximation) -----
```

```
> system.time( meig_test <- meigen( coords = coords_test ) )
```

user	system	elapsed
242.28	1.44	243.79

```

-----CP time (with approximation) -----
> system.time( meig_test200 <- meigen_f( coords = coords_test )
  user  system  elapsed
  0.37   0.00   0.38
> system.time( meig_test100 <- meigen_f( coords = coords_test, enum = 100 ) )
  user  system  elapsed
  0.15   0.00   0.16
> system.time( meig_test50 <- meigen_f( coords = coords_test, enum = 50 ) )
  user  system  elapsed
  0.08   0.00   0.08

```

Figure 3 maps the calculated 1st, 10th, and 100th eigenvectors. It is important to note that, while approximated and exact eigenvectors can have different map patterns respectively, both of them describe patterns in similar spatial scales. In other words, in both cases, 1st eigenvectors describe global map patterns, 10th medium-scale patterns, and 100th local patterns.

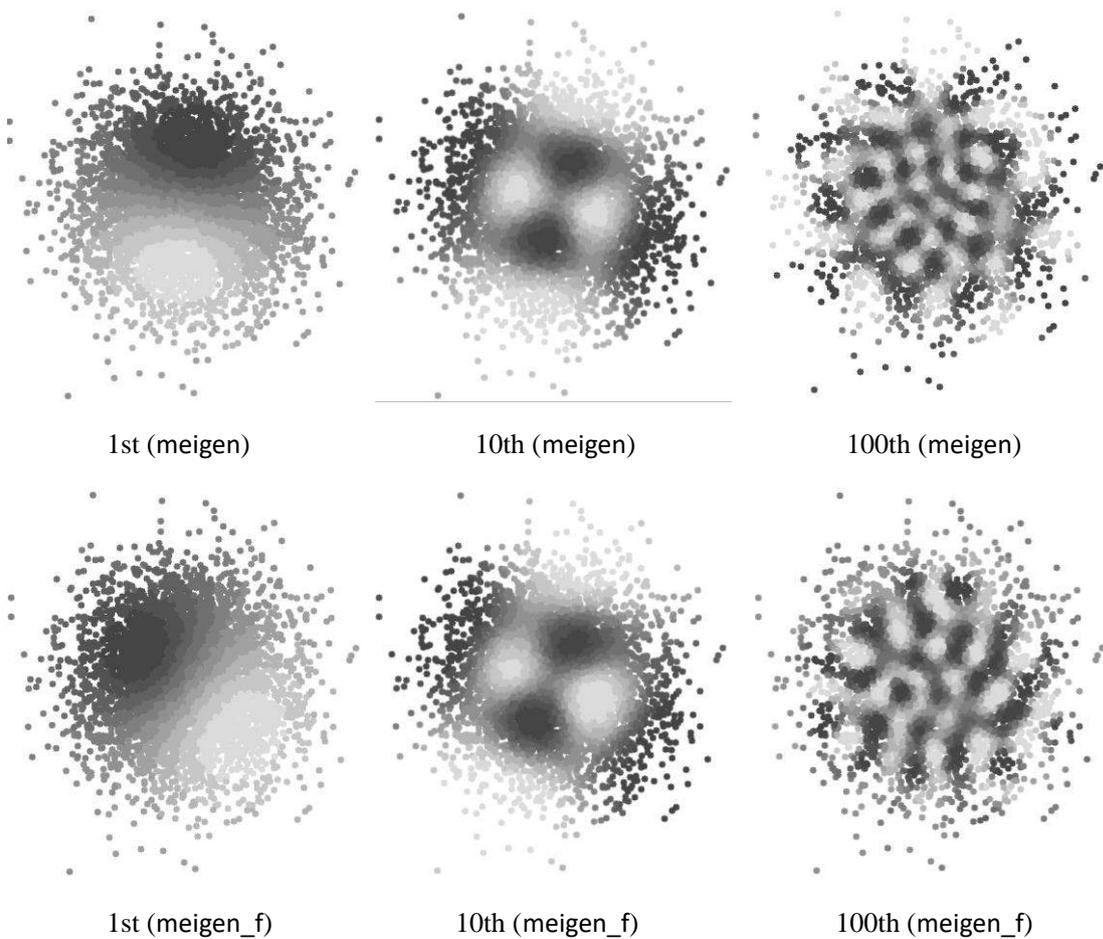


Figure 3. The 1st, 10th, and 100th eigenvectors extracted from meigen and meigen_f

5.2. Parameter estimation

The basic ESF model is estimated computationally efficiently by setting `fn = "all"` in the function `esf`. The RE-ESF model is estimated by small computational cost by the function `resf`, by default.

The RE-ESF-based SVC model can also be estimated computationally efficiently by the `resf_vc` function. To achieve this, (i) the number of eigen-pairs in `meig` must not be large. It is achieved by setting `enum = 200` that is sufficiently small and approximation error is sufficiently small (Murakami and Griffith, 2017). Besides, the number of SVCs must also be small. It is fulfilled by including at most about 4 explanatory variables into `x`, and the others into `xconst`. After all, the following command implements the SVC model computationally efficiently:

```
-----Eigen-decomposition -----
> meig  <- meigen( coords = coords, enum = 200)    # slow, but exact
or alternatively,
> meig  <- meigen_f( coords = coords )            # fast, but approximation

-----Parameter estimation -----
> xv    <- x[ , c( "x1", "x2", "x3", "x4" ) ]      # at most about 4 explanatory variables
> xconst <- x[ , c( "x5", "x6", "x7", "x8", "x9", "x10" ) ] # the other explanatory variables
> rv_res <- resf_vc( y = y, x = xv, xconst = xconst, meig = meig, method = "reml" )
```

The SF-UQR model requires a bootstrapping to estimate confidential intervals for the coefficients. However, computational cost for the iteration does not depend on sample size, but only on the number of eigenvectors in `meig` (see, Murakami and Seya, 2017). That is, the SF-UQR is applicable to large data if only `meig` is defined just as mentioned above.

6. Future directions

I plan to enrich functions relating Moran's eigenvector-based regression approach gradually.

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