Spatial regression using the spmoran package: Boston housing price data examples

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Contents

1 Introduction

This package provides functions for estimating Gaussian and non-Gaussian spatial regression models and extensions, including spatially and non-spatially varying coefficient models, models with group effects, spatial unconditional quantile regression models, and low rank spatial econometric models. All these models are estimated computationally efficiently.

An approximate Gaussian process (GP or kriging model), which is interpretable in terms of the Moran coefficient (MC), is used for modeling the spatial process. The approximate GP is defined by a linear combination of the Moran eigenvectors (MEs) corresponding to positive eigenvalue, which are known to explain positive spatial dependence. The resulting spatial process describes positively dependent map patterns (i.e., MC > 0), which are dominant in regional science (Griffith, 2003). Below, the spmoran package is used to analyze the Boston housing dataset.

The sample codes used below are available from https://github.com/dmuraka/spmoran.

```
library(spmoran)
```

2 Gaussian spatial additive mixed models

2.1 Basic models

This section considers the following model:

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_k + f_{MC}(s_i) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma^2),$$

which decomposes the explained variable y_i observed at the i-th sample site into trend $\sum_{k=1}^{K} x_{i,k}\beta_{i,k}$, spatial process $f_{MC}(s_i)$ depending on location s_i, and noise ϵ_i. The spatial process is required to eliminate residual spatial dependence and estimate/infer regression coefficients β_k appropriately. ESF and RE-ESF define $f_{MC}(s_i)$ using the MC-based spatial process to efficiently eliminate residual spatial dependence. These processes are defined by the weighted sum of the Moran eigenvectors (MEs), which are spatial basis functions (distinct map pattern variables; see Griffith, 2003).

2.1.1 Eigenvector spatial filtering (ESF)

ESF specifies $f_{MC}(s_i)$ using an MC-based deterministic spatial process (see Griffith, 2003). Below is a code estimating the linear ESF model. In the code, the meigen function extracts the MEs, and the esf function estimates the model.

```
require(spdep)
data(boston)
y       <- boston.c[, "CMEDV" ]
x       <- boston.c[,c("CRIM","ZN","INDUS", "CHAS", "NOX","RM", "AGE")]
coords<- boston.c[,c("LON","LAT")]

#########Distance-based ESF
meig    <- meigen(coords=coords)
res    <- esf(y=y,x=x,meig=meig, vif=10)
res
```

```
## Call:
## esf(y = y, x = x, vif = 10, meig = meig)
```

```
##
## ----Coefficients------------------------------
##                  Estimate         SE    t_value      p_value
## (Intercept)  11.34040959 3.91692274  2.8952344 3.968277e-03
## CRIM         -0.20942091 0.03048530 -6.8695702 2.089395e-11
## ZN            0.02322000 0.01384823  1.6767492 9.426799e-02
## INDUS        -0.15063613 0.06823776 -2.2075188 2.776856e-02
## CHAS          0.15172838 0.93842988  0.1616832 8.716260e-01
## NOX         -38.02167637 4.79403898 -7.9310320 1.651338e-14
## RM            6.33316024 0.36887955 17.1686403 1.842211e-51
## AGE          -0.07820247 0.01564970 -4.9970593 8.274067e-07
##
## ----Spatial effects (residuals)----------------
##                       Estimate
## SE                   6.8540461
## Moran.I/max(Moran.I) 0.6701035
##
## ----Error statistics--------------------------
##                   stat
## resid_SE      4.476459
## adjR2         0.762328
## logLik    -1453.376154
## AIC        2996.752308
## BIC        3186.946458
```

While the meigen function is slow for large samples, it can be substituted with the meigen_f function performing a fast eigen-approximation. Here is a fast ESF code for large samples:

```
meig_f<- meigen_f(coords)
res   <- esf(y=y, x=x, meig=meig_f,vif=10, fn="all")
```

2.1.2 Random effects ESF (RE-ESF)

RE-ESF specifies $f_{MC}(s_i)$ using an MC-based spatial random process, again to eliminate residual spatial dependence (see Murakami and Griffith, 2015). Here is a sample example:

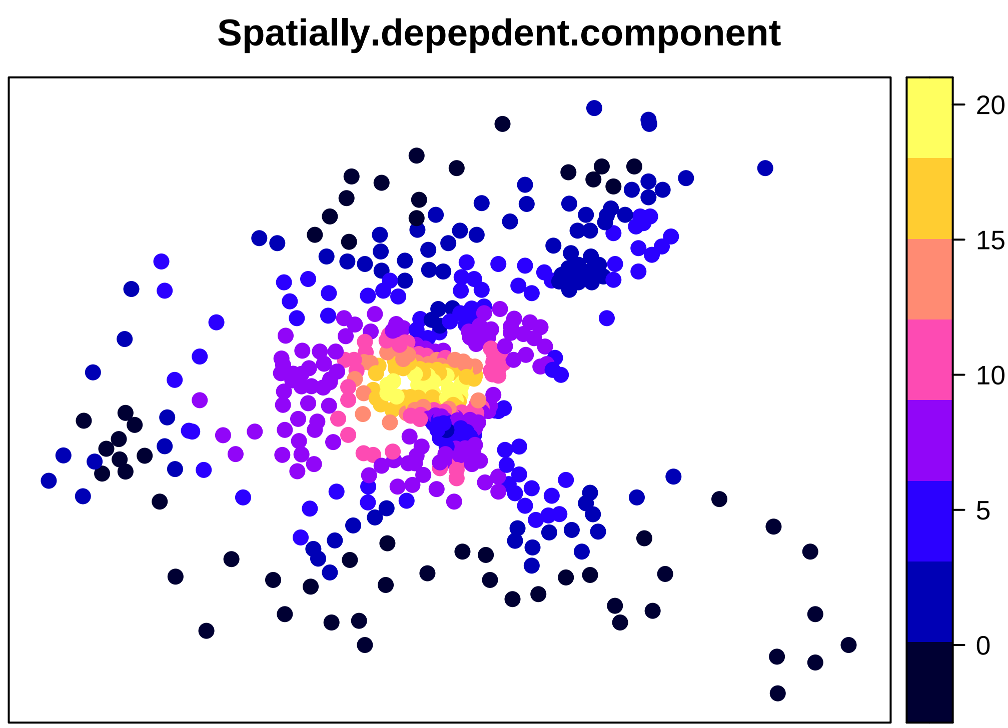
```
res   <- resf(y = y, x = x, meig = meig)
res
```

```
## Call:
## resf(y = y, x = x, meig = meig)
##
## ----Coefficients------------------------------
##                  Estimate         SE    t_value      p_value
## (Intercept)   6.63220350 3.94484193  1.6812343 9.340107e-02
## CRIM         -0.19815203 0.03126666 -6.3374866 5.608678e-10
## ZN            0.01453736 0.01591772  0.9132814 3.615764e-01
## INDUS        -0.15560251 0.06842940 -2.2739131 2.343446e-02
## CHAS          0.51046251 0.92329946  0.5528678 5.806245e-01
## NOX         -31.26690020 5.02069123 -6.2276087 1.075126e-09
## RM            6.33993146 0.36671337 17.2885202 0.000000e+00
## AGE          -0.06351412 0.01526957 -4.1595218 3.810682e-05
##
## ----Variance parameter------------------------
##
```

```
## Spatial effects (residuals):
##                        (Intercept)
## random_SE                6.7424433
## Moran.I/max(Moran.I)     0.6648678
##
## ----Error statistics-------------------------
##                        stat
## resid_SE         4.3515211
## adjR2(cond)      0.7735912
## rlogLik      -1540.3812428
## AIC           3102.7624855
## BIC           3149.2543889
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

The residual spatial process $f_{MC}(s_i)$ is plotted as follows:

```
plot_s(res)
```



For large data, the meigen_f function is available again:

```
meig_f<- meigen_f(coords)
res   <- resf(y = y, x = x, meig = meig_f)
```

The meigen_f function is available for all the regression models explained below.

2.2 Extended models

2.2.1 Models with non-spatially varying coefficients (coefficients varying wrt covariate value)

Influence from covariates can vary depending on covariate value. For example, distance to railway station might have a strong impact on housing price if the distance is small, while it might be weak if the distance is large. To capture such an effect, the resf function estimates coefficients varying with respect to covariate value. I call such coefficients non-spatially varying coefficients (NVCs). If nvc=TRUE, the resf function estimates the following model considering NSVs and residual spatial dependence:

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_{i,k} + f_{MC}(s_i) + \epsilon_i, \qquad \beta_{i,k} = b_k + f(x_{i,k}), \qquad \epsilon_i \sim N(0, \sigma^2),$$

where $f(x_{i,k})$ is a smooth function of $x_{i,k}$ capturing the non-spatial influence. Here is a code estimating a spatial NVC model (with selection of constant or NVC):

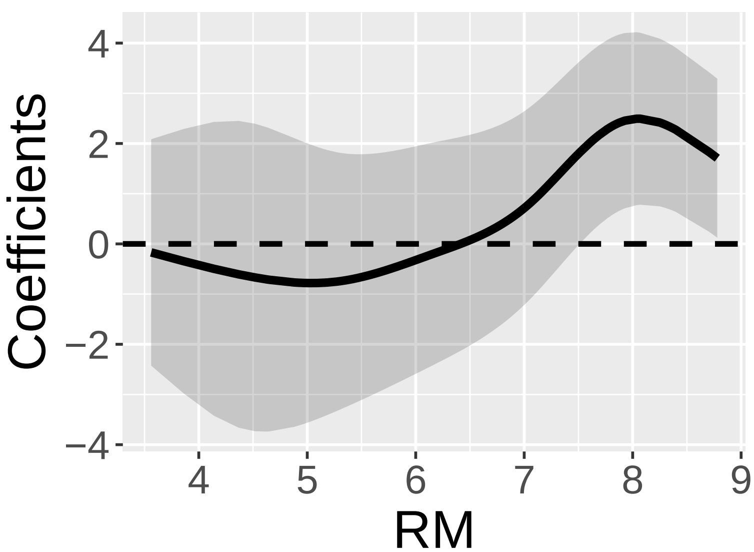
```
res   <- resf(y = y, x = x, meig = meig, nvc=TRUE)
res
```

```
## Call:
## resf(y = y, x = x, nvc = TRUE, meig = meig)
##
## ----Non-spatially varying coefficients on x (summary) ----
##
## Coefficients:
##    Intercept          CRIM              ZN               INDUS
##  Min.   :25.41   Min.   :-0.1822   Min.   :0.02042   Min.   :-0.2119
##  1st Qu.:25.41   1st Qu.:-0.1822   1st Qu.:0.02042   1st Qu.:-0.2119
##  Median :25.41   Median :-0.1822   Median :0.02042   Median :-0.2119
##  Mean   :25.41   Mean   :-0.1822   Mean   :0.02042   Mean   :-0.2119
##  3rd Qu.:25.41   3rd Qu.:-0.1822   3rd Qu.:0.02042   3rd Qu.:-0.2119
##  Max.   :25.41   Max.   :-0.1822   Max.   :0.02042   Max.   :-0.2119
##       CHAS            NOX               RM                AGE
##  Min.   :1.375   Min.   :-0.463   Min.   :-0.78043   Min.   :-0.06742
##  1st Qu.:1.375   1st Qu.: 6.083   1st Qu.:-0.40834   1st Qu.:-0.06742
##  Median :1.375   Median : 7.792   Median :-0.16098   Median :-0.06742
##  Mean   :1.375   Mean   : 7.074   Mean   : 0.03975   Mean   :-0.06742
##  3rd Qu.:1.375   3rd Qu.: 8.654   3rd Qu.: 0.19417   3rd Qu.:-0.06742
##  Max.   :1.375   Max.   :11.517   Max.   : 2.49406   Max.   :-0.06742
##
## Statistical significance:
##                            Intercept CRIM  ZN INDUS CHAS NOX  RM AGE
## Not significant                    0    0 506     0    0 506 472   0
## Significant (10% level)            0    0   0     0  506   0   7   0
## Significant ( 5% level)            0    0   0     0    0   0  10   0
## Significant ( 1% level)          506  506   0   506    0   0  17 506
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
##                       (Intercept)
## random_SE               3.6981527
## Moran.I/max(Moran.I)    0.4490228
##
## Non-spatial effects (coefficients on x):
```

```
##            CRIM ZN INDUS CHAS      NOX        RM AGE
## random_SE    0  0     0    0 1.850518 0.2459548   0
##
## ----Error statistics-------------------------
##                       stat
## resid_SE        3.7949128
## adjR2(cond)     0.8271073
## rlogLik     -1478.6128728
## AIC          2983.2257457
## BIC          3038.1707224
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

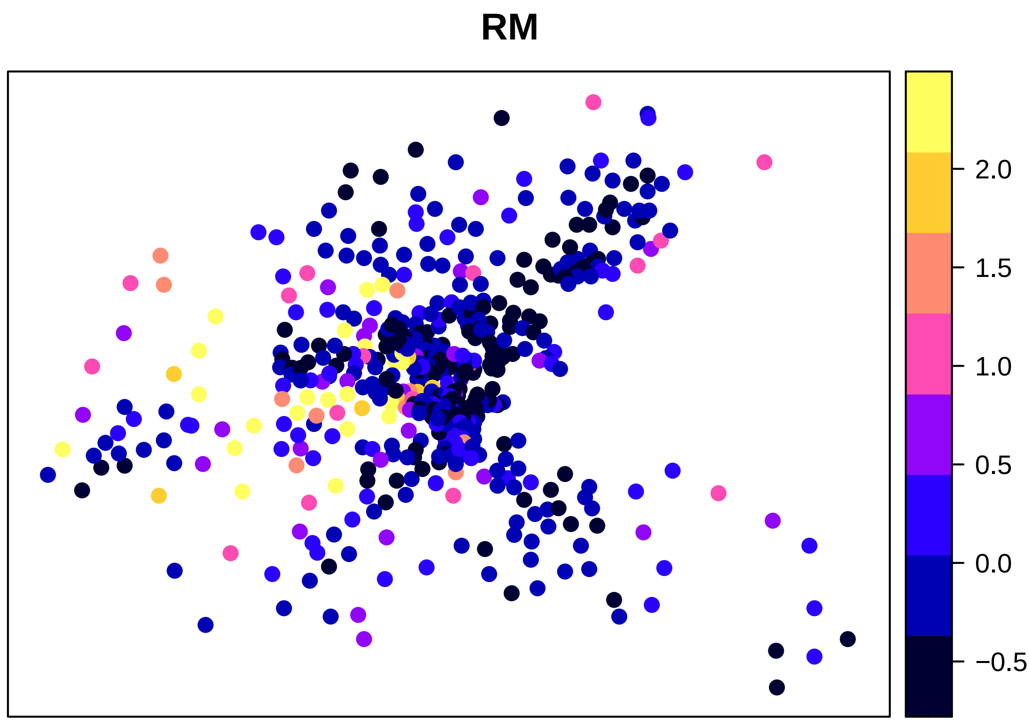
By default, this function selects constant or NVC through BIC minimization. "Non-spatially varying coefficients" in the "Variance parameter" section summarizes the estimated standard errors of the NVCs. Based on the result, coefficients on {NOX, RM} are NVCs, and coefficients on the others are constants. The NVC on RM, which is the 6-th covariate, is plotted as below. The solid line in the panel denotes the estimated NVC, and the gray area denotes the 95% confidence interval. This plot shows that RM is positively statistically significant only if RM is large.

```r
plot_n(res,6)
```



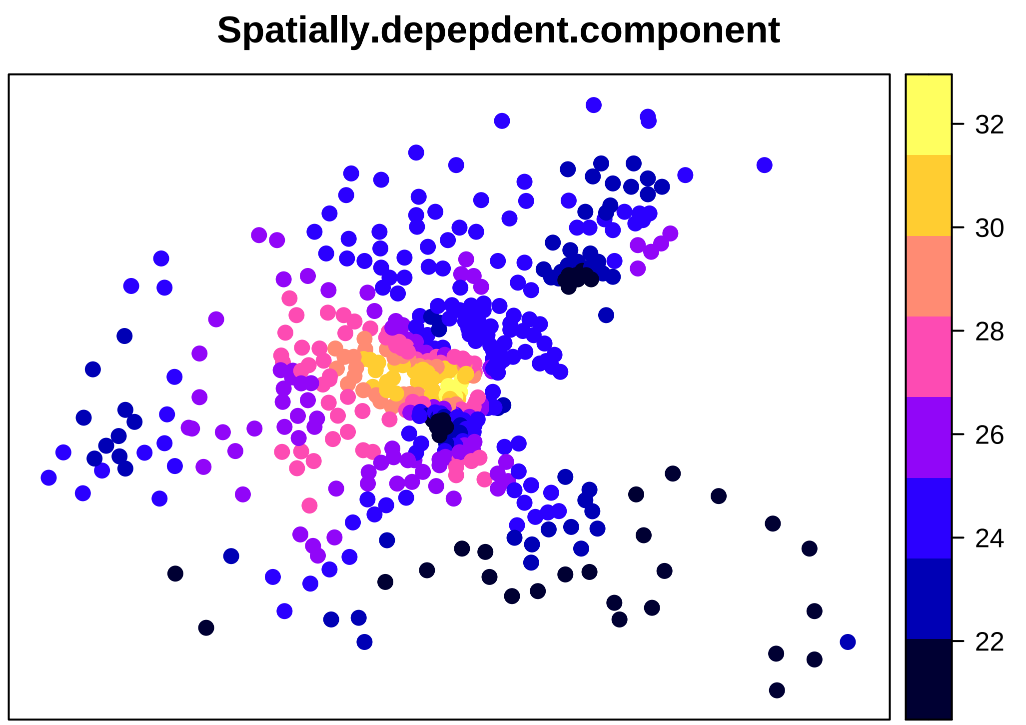
The NVC can also be spatially plotted as below:

```r
plot_s(res,6)
```



On the other hand, the residual spatial process $f_{MC}(s_i)$ is plotted as

```
plot_s(res)
```



Sometimes, the user may wish to assume NVCs only on the first three covariates and constant coefficients on the others. The following code estimates such a model:

```
res   <- resf(y = y, x = x, meig = meig, nvc=TRUE, nvc_sel=1:3)
```

2.2.2 Models with spatially varying coefficients

This package implements an ME-based spatially varying coefficient (M-SVC) model (Murakami et al., 2017), which is formulated as

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_{i,k} + f_{MC}(s_i) + \epsilon_i, \qquad \beta_{i,k} = b_k + f_{MC,k}(s_i), \qquad \epsilon_i \sim N(0, \sigma^2),$$

This model defines the k-th coefficient at site i by $\beta_{i,k}$= [constant mean b_k] + [spatially varying component $f_{MC,k}(s_i)$]. Geographically weighted regression (GWR) is known as another SVC estimation approach. Major advantages of the M-SVC modeling approach over GWR are as follows:

- The M-SVC model estimates the spatial scale (or MC value) of each SVC, while the classical GWR assumes a common scale across SVCs.
- The M-SVC model can assume SVCs on some covariates and constant coefficients on the others. This is achieved by simply assuming $\beta_{i,k} = b_k$
- This model is faster and available for very large samples. In addition, the model is free from memory limitations if the besf_vc function is used (see Section 4).
- Model selection (i.e., constant coefficient or SVC) is implemented without losing its computational efficiency.

Here is a sample code estimating an SVC model without coefficient type selection. In the code, x specifies covariates assuming SVCs, while xconst specifies covariates assuming constant coefficients. If x_sel = FALSE, the types of coefficients on x are fixed.

```
y         <- boston.c[, "CMEDV"]
x       <- boston.c[,c("CRIM", "AGE")]
xconst  <- boston.c[,c("ZN","DIS","RAD","NOX",  "TAX","RM", "PTRATIO", "B")]
coords  <- boston.c[,c("LON","LAT")]
meig      <- meigen(coords=coords)
res     <- resf_vc(y=y,x=x,xconst=xconst,meig=meig, x_sel = FALSE )
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/3"
## [1] "2/3"
## [1] "3/3"
## [1] "BIC: 3120.605"
## [1] "-------  Iteration 2  -------"
## [1] "1/3"
## [1] "2/3"
## [1] "3/3"
## [1] "BIC: 3114.252"
## [1] "-------  Iteration 3  -------"
## [1] "1/3"
## [1] "2/3"
## [1] "3/3"
## [1] "BIC: 3114.139"
## [1] "-------  Iteration 4  -------"
## [1] "1/3"
## [1] "2/3"
## [1] "3/3"
## [1] "BIC: 3114.138"
```

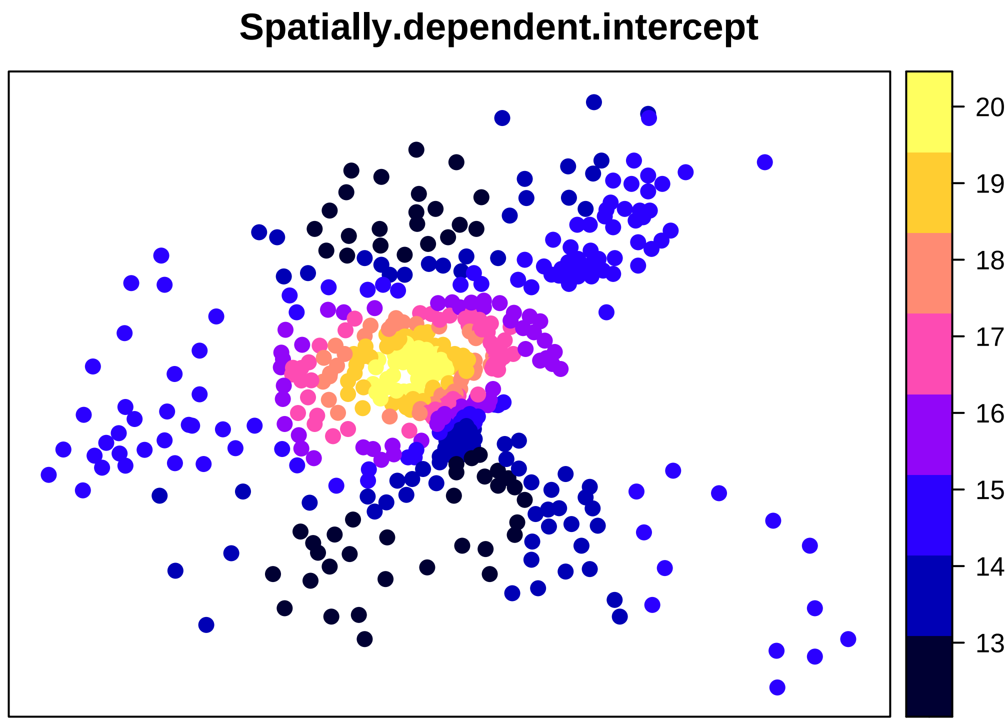
```
res
```

```
## Call:
## resf_vc(y = y, x = x, xconst = xconst, x_sel = FALSE, meig = meig)
```

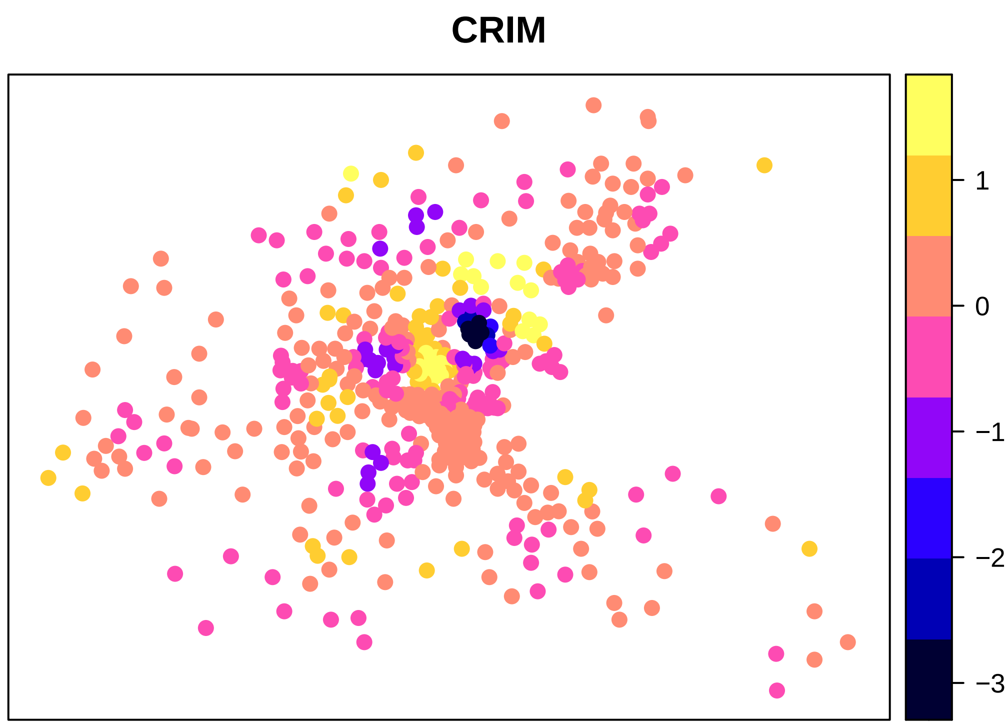
```
##
## ----Spatially varying coefficients on x (summary)----
##
## Coefficient estimates:
##   (Intercept)         CRIM                AGE
##  Min.   :12.03   Min.   :-3.29294   Min.   :-0.14986
##  1st Qu.:13.99   1st Qu.:-0.19941   1st Qu.:-0.08377
##  Median :15.06   Median : 0.04993   Median :-0.06780
##  Mean   :15.70   Mean   : 0.05902   Mean   :-0.06582
##  3rd Qu.:17.31   3rd Qu.: 0.36587   3rd Qu.:-0.04710
##  Max.   :20.46   Max.   : 1.83866   Max.   : 0.04298
##
## Statistical significance:
##                            Intercept CRIM AGE
## Not significant                    0  416 147
## Significant (10% level)            0   27  40
## Significant ( 5% level)          190   17  99
## Significant ( 1% level)          316   46 220
##
## ----Constant coefficients on xconst----------------------------
##             Estimate          SE   t_value      p_value
## ZN        0.03202068 0.013219003  2.422322 1.582817e-02
## DIS      -1.47514930 0.334360238 -4.411856 1.292875e-05
## RAD       0.36064288 0.090818317  3.971037 8.368693e-05
## NOX     -36.21088316 5.134427150 -7.052565 6.925571e-12
## TAX      -0.01242296 0.003502523 -3.546862 4.320840e-04
## RM        6.49212566 0.326197980 19.902409 0.000000e+00
## PTRATIO  -0.52573980 0.151594626 -3.468064 5.762765e-04
## B         0.02091202 0.003094117  6.758638 4.477529e-11
##
## ----Variance parameters----------------------------------
##
## Spatial effects (coefficients on x):
##                        (Intercept)       CRIM        AGE
## random_SE                3.9039832 1.59443322 0.05746111
## Moran.I/max(Moran.I)     0.6627375 0.04502003 0.06267778
##
## ----Error statistics-------------------------------------
##                        stat
## resid_SE          3.6706778
## adjR2(cond)       0.8375658
## rlogLik       -1501.0302460
## AIC            3038.0604921
## BIC            3114.1381521
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x and xconst)
```

Estimated SVCs can be plotted as

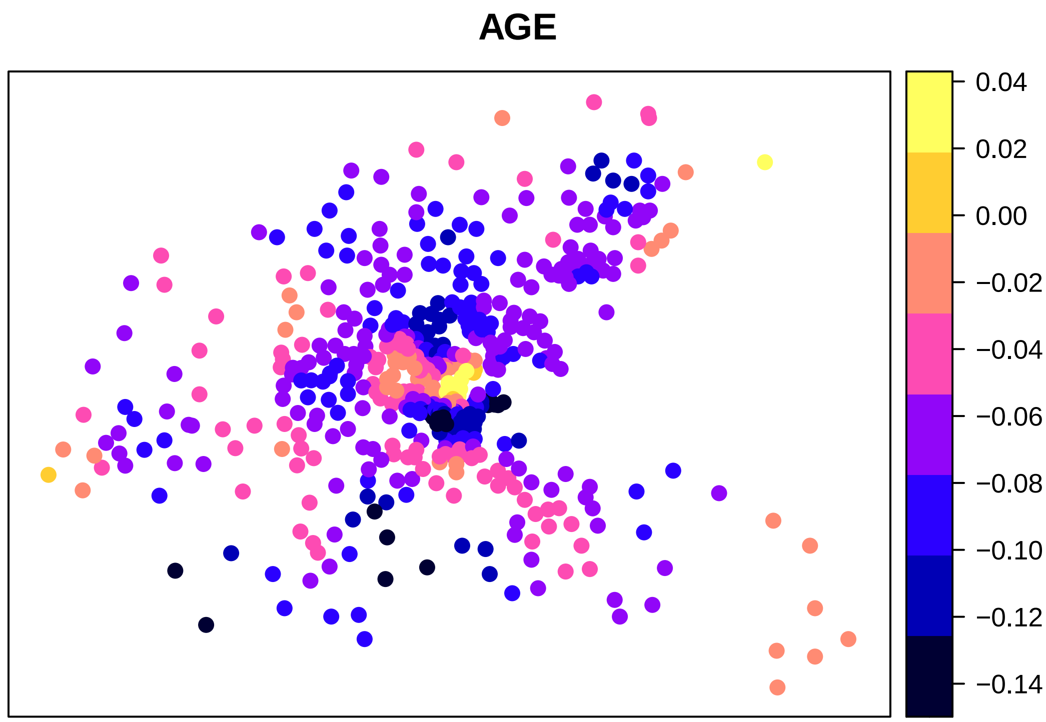
```
plot_s(res,0) # Spatially varying intercept
```




```r
plot_s(res,1) # 1st SVC
```




```r
plot_s(res,2) # 2nd SVC
```



On the other hand, by default, the resf_vc function selects constant or SVCs through a BIC minimization (i.e., x_sel=TRUE by default). Here is a code:

```
res     <- resf_vc(y=y,x=x,xconst=xconst,meig=meig )
```

2.2.3 Models with spatially and non-spatially varying coefficients

The spatially and non-spatially varying coefficient (SNVC) model is defined as

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_{i,k} + f_{MC}(s_i) + \epsilon_i, \qquad \beta_{i,k} = b_k + f_{MC,k}(s_i) + f(x_{i,k}), \qquad \epsilon_i \sim N(0, \sigma^2),$$

This model defines the k-th coefficient as $\beta_{i,k}$= [constant mean b_k] + [spatially varying component $f_{MC,k}(s_i)$] + [non-spatially varying component $f(x_{i,k})$]. Murakami and Griffith (2020) showed that, unlike SVC models that tend to be unstable owing to spurious correlation among SVCs (see Wheeler and Tiefelsdorf, 2005), this SNVC model is stable and quite robust against spurious correlations. Therefore, I recommend using the SNVC model, even if the purpose of the analysis is estimating SVCs.

An SNVC model is estimated by specifying x_nvc = TRUE in the resf_vc function as follows:

```
res   <- resf_vc(y=y,x=x,xconst=xconst,meig=meig, x_nvc =TRUE)
```

This model assumes SNVC on x and constant coefficients on xconst. By default, the coefficient type (SNVC, SVC, NVC, or constant) on x is selected.

It is also possible to assume SNVCs on x and NVCs on xcnost by specifying xconst_nvc = TRUE as follows:

```
res   <- resf_vc(y=y,x=x,xconst=xconst,meig=meig, x_nvc =TRUE, xconst_nvc=TRUE)
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/13"
## [1] "2/13"
## [1] "3/13"
```

```
## [1] "4/13"
## [1] "5/13"
## [1] "7/13"
## [1] "8/13"
## [1] "9/13"
## [1] "10/13"
## [1] "11/13"
## [1] "12/13"
## [1] "13/13"
## [1] "BIC: 3023.362"
## [1] "-------  Iteration 2  -------"
## [1] "1/13"
## [1] "2/13"
## [1] "3/13"
## [1] "4/13"
## [1] "5/13"
## [1] "7/13"
## [1] "8/13"
## [1] "9/13"
## [1] "10/13"
## [1] "11/13"
## [1] "12/13"
## [1] "13/13"
## [1] "BIC: 3013.007"
## [1] "-------  Iteration 3  -------"
## [1] "1/13"
## [1] "2/13"
## [1] "3/13"
## [1] "4/13"
## [1] "5/13"
## [1] "7/13"
## [1] "8/13"
## [1] "9/13"
## [1] "10/13"
## [1] "11/13"
## [1] "12/13"
## [1] "13/13"
## [1] "BIC: 3012.859"
## [1] "-------  Iteration 4  -------"
## [1] "1/13"
## [1] "2/13"
## [1] "3/13"
## [1] "4/13"
## [1] "5/13"
## [1] "7/13"
## [1] "8/13"
## [1] "9/13"
## [1] "10/13"
## [1] "11/13"
## [1] "12/13"
## [1] "13/13"
## [1] "BIC: 3012.857"
## [1] "-------  Iteration 5  -------"
## [1] "1/13"
```

```
## [1] "2/13"
## [1] "3/13"
## [1] "4/13"
## [1] "5/13"
## [1] "7/13"
## [1] "8/13"
## [1] "9/13"
## [1] "10/13"
## [1] "11/13"
## [1] "12/13"
## [1] "13/13"
## [1] "BIC: 3012.857"
```

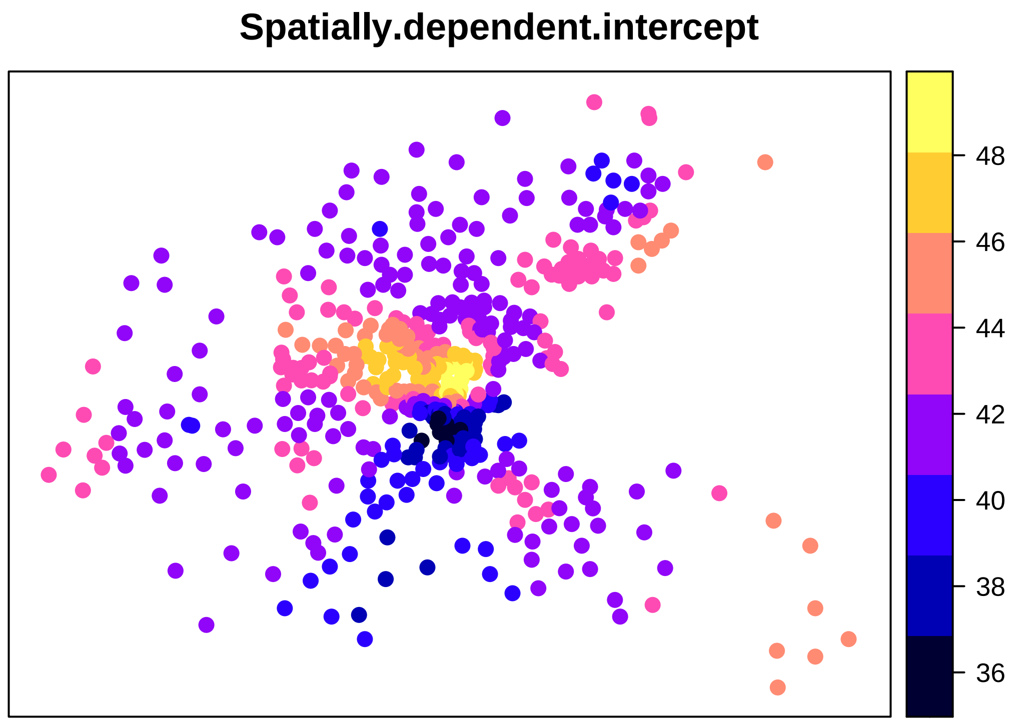
```
res
```

```
## Call:
## resf_vc(y = y, x = x, xconst = xconst, x_nvc = TRUE, xconst_nvc = TRUE,
##     meig = meig)
##
## ----Spatially and non-spatially varying coefficients on x (summary)----
##
## Coefficient estimates:
##   (Intercept)         CRIM              AGE
##  Min.   :34.97   Min.   :-2.1712   Min.   :-0.07496
##  1st Qu.:40.94   1st Qu.:-0.6141   1st Qu.:-0.07496
##  Median :42.28   Median :-0.4156   Median :-0.07496
##  Mean   :42.43   Mean   :-0.4288   Mean   :-0.07496
##  3rd Qu.:43.77   3rd Qu.:-0.2156   3rd Qu.:-0.07496
##  Max.   :49.94   Max.   : 0.5235   Max.   :-0.07496
##
## Statistical significance:
##                         Intercept CRIM AGE
## Not significant                 0  394   0
## Significant (10% level)         0   15   0
## Significant ( 5% level)         0   29   0
## Significant ( 1% level)       506   68 506
##
## ----Non-spatially varying coefficients on xconst (summary)----
##
## Coefficient estimates:
##        ZN                DIS               RAD              NOX
##  Min.   :0.02511   Min.   :-1.107   Min.   :0.6289   Min.   :-23.31
##  1st Qu.:0.02511   1st Qu.:-1.107   1st Qu.:0.6289   1st Qu.:-19.39
##  Median :0.02511   Median :-1.107   Median :0.6289   Median :-18.49
##  Mean   :0.02511   Mean   :-1.107   Mean   :0.6289   Mean   :-18.56
##  3rd Qu.:0.02511   3rd Qu.:-1.107   3rd Qu.:0.6289   3rd Qu.:-17.58
##  Max.   :0.02511   Max.   :-1.107   Max.   :0.6289   Max.   :-14.48
##       TAX                 RM             PTRATIO              B
##  Min.   :-0.01512   Min.   :0.6017   Min.   :-0.6371   Min.   :0.01371
##  1st Qu.:-0.01512   1st Qu.:0.8399   1st Qu.:-0.6371   1st Qu.:0.01371
##  Median :-0.01512   Median :1.0419   Median :-0.6371   Median :0.01371
##  Mean   :-0.01512   Mean   :1.2079   Mean   :-0.6371   Mean   :0.01371
##  3rd Qu.:-0.01512   3rd Qu.:1.3036   3rd Qu.:-0.6371   3rd Qu.:0.01371
##  Max.   :-0.01512   Max.   :3.2998   Max.   :-0.6371   Max.   :0.01371
##
```

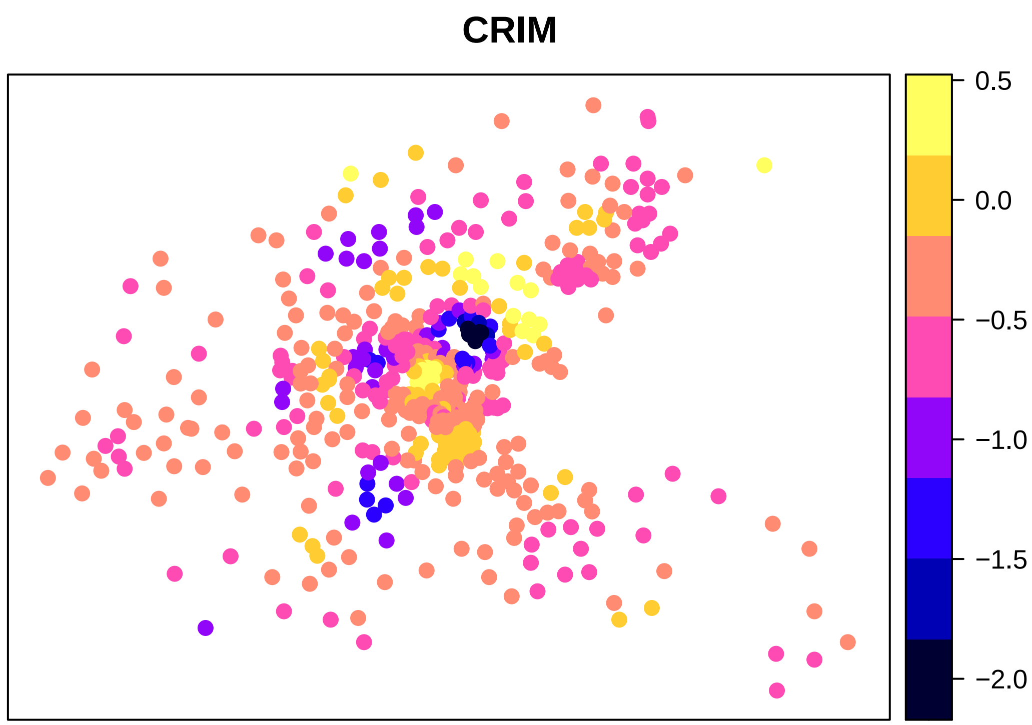
```
## Statistical significance:
##                          ZN DIS RAD NOX TAX  RM PTRATIO   B
## Not significant           0   0   0 185   0 414       0   0
## Significant (10% level) 506   0   0 217   0  27       0   0
## Significant ( 5% level)   0   0   0  40   0  23       0   0
## Significant ( 1% level)   0 506 506  64 506  42     506 506
##
## ----Variance parameters---------------------------------
##
## Spatial effects (coefficients on x):
##                      (Intercept)      CRIM AGE
## random_SE              4.0667763 1.0007384   0
## Moran.I/max(Moran.I)   0.3274953 0.0743859  NA
##
## Non-spatial effects (coefficients on x):
##                  CRIM AGE
## random_SE 0.03405477   0
##
## Non-spatial effects (coefficients on xconst):
##           ZN DIS RAD      NOX TAX       RM PTRATIO B
## random_SE  0   0   0 1.496402   0 0.200113       0 0
##
## ----Error statistics-------------------------------------
##                      stat
## resid_SE        3.1945901
## adjR2(cond)     0.8767156
## rlogLik     -1447.2763228
## AIC          2932.5526456
## BIC          3012.8568423
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x and xconst)
```

By default, the coefficient type (SNVC, SVC, NVC, or constant) on x and those (NVC or const) on xconst are selected. The estimated SNVCs are plotted as follows:

```r
plot_s(res,0)            # Spatially varying intercept
```

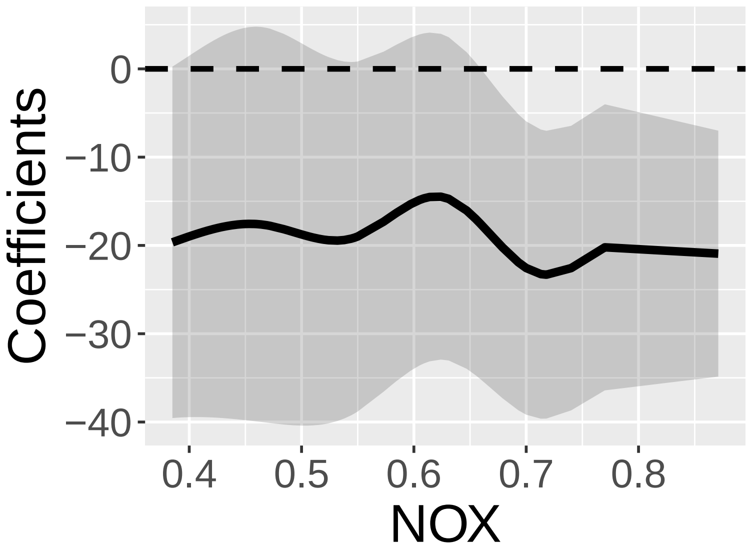



```
plot_s(res,1)            # SNVC on x[,1]
```

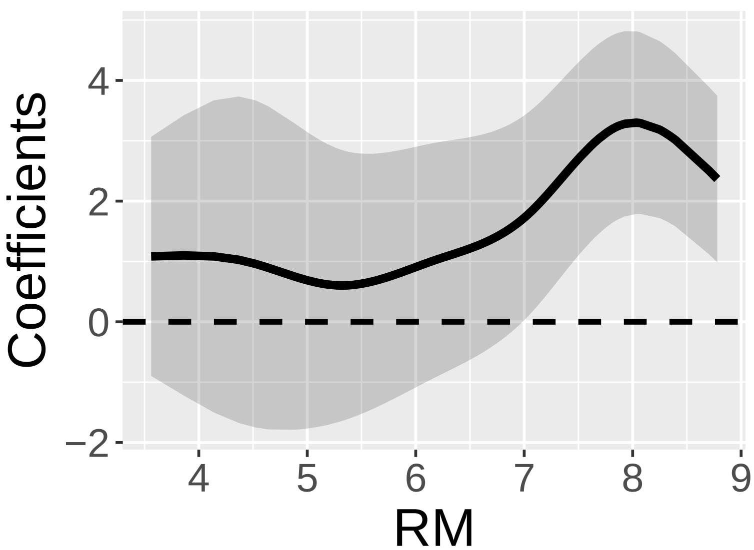


NVCs on xconst is plotted by specifying xtype="xconst" in the plot_n function, as below. The solid line denotes the estimated NVC, and the gray area denotes the 95% confidence interval:

```
plot_n(res,4,xtype="xconst")#NVC on xconst[,4]
```

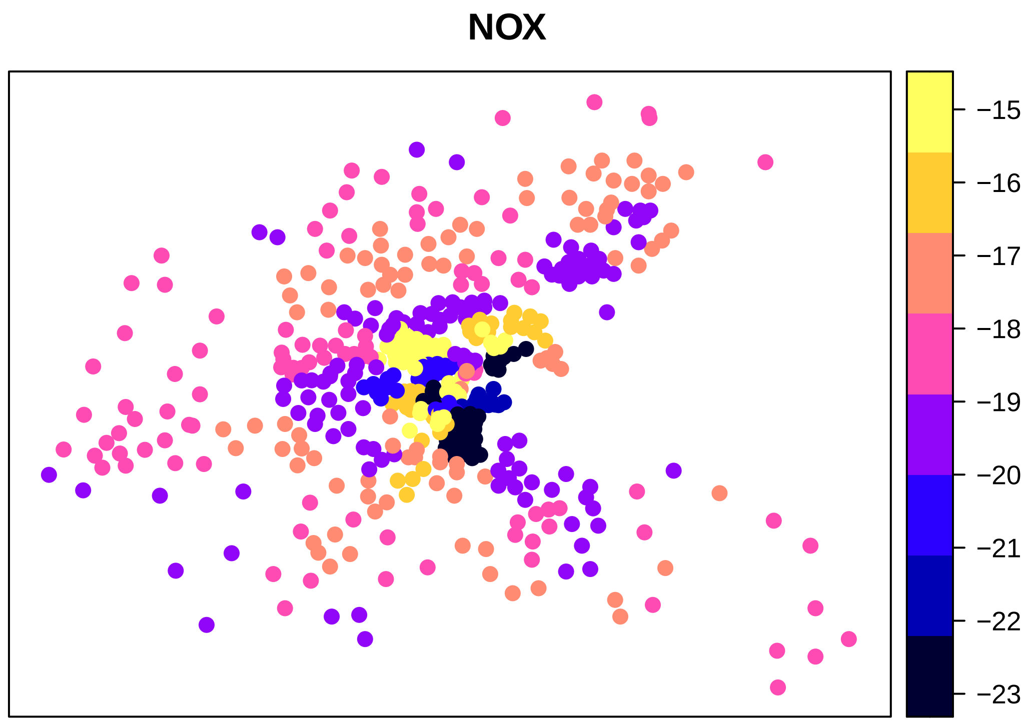



```
plot_n(res,6,xtype="xconst")#NVC on xconst[,6]
```

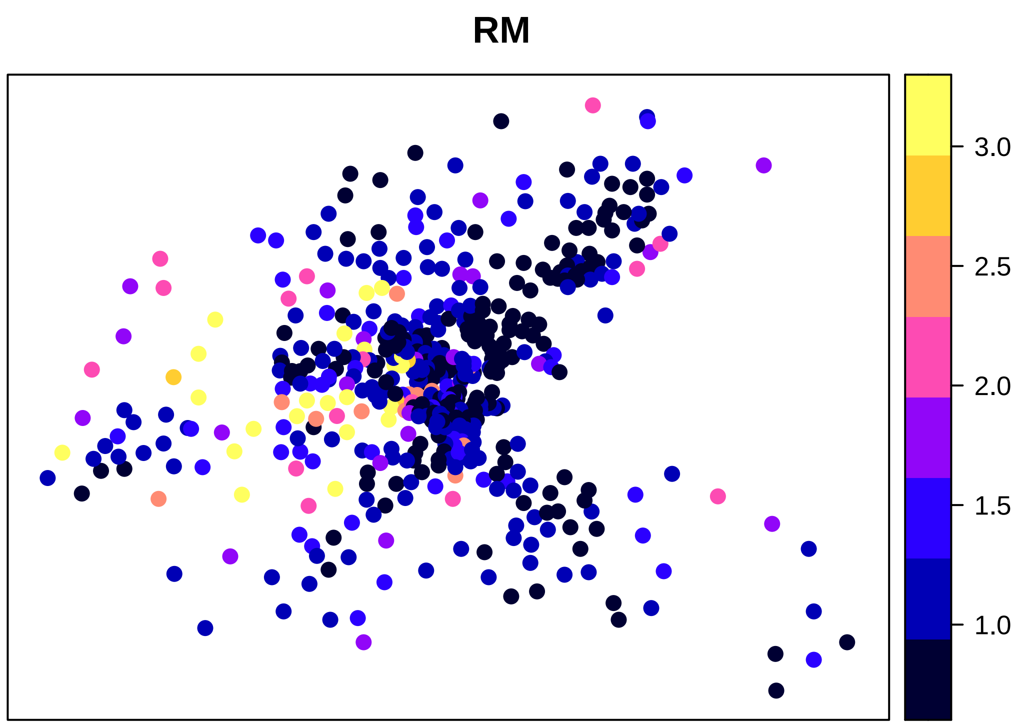


These NVCs can also be plotted spatially as follows:

```
plot_s(res,4,xtype="xconst")#Spatial plot of NVC on xconst[,4]
```




```
plot_s(res,6,xtype="xconst")#Spatial plot of NVC on xconst[,6]
```



2.2.4 Models with group effects

2.2.4.1 Outline Two group effects are available in this package:

1. Spatially dependent group effects. Spatial dependence among groups is modeled instead of modeling spatial dependence among individuals.

2. Spatially independent group effects assuming independence across groups (usual group effects)

They are estimated in the resf and resf_vc functions. When considering both these effects, the resf function estimates the following model (if no NVC is assumed):

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_k + f_{MC}(g_{I(0)}) + \sum_{h=1}^{H} \gamma(g_{I(h)}) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma^2),$$

where $g_{I(0)}, g_{I(1)}, \ldots, g_{I(H)}$ represent group variables. $f_{MC}(g_{I(0)})$ denotes spatially dependent group effects, while $\gamma(g_{I(h)})$ denotes spatially independent group effects for the h-th group variable. On the other hand, the resf_vc function can estimate the following model considering these two effects (again, no NVC is assumed):

$$y_i = \sum_{k=1}^{K} x_{i,k}\beta_{i,k} + f_{MC}(g_{I(0)}) + \sum_{h=1}^{H} \gamma(g_{I(h)}) + \epsilon_i, \qquad \beta_{i,k} = b_k + f_{MC,k}(g_{i(0)}), \qquad \epsilon_i \sim N(0, \sigma^2),$$

Below, multilevel modeling, small area estimation, and panel data analysis are demonstrated.

2.2.4.2 Multilevel model Data often have a multilevel structure. For example, the school achievement of individual students changes depending on the class and school. A condominium unit price depends, not only on unit attributes, but also on building attributes. Multilevel modeling is required to explicitly consider the multilevel structure behind data and perform spatial regressions.

This section demonstrates the modeling considering the two group effects using the resf function. The data used are the Boston housing datasets that consist of 506 samples in 92 towns, which are regarded as groups. To model spatially dependent group effects, Moran eigenvectors are defined by groups. This is done by specifying s_id in the meigen function using a group variable, which is the town name (TOWNNO), in this case, as follows:

```
xgroup<- boston.c[,"TOWNNO"]
coords<- boston.c[,c("LON","LAT")]
meig_g<- meigen(coords=coords, s_id=xgroup)
```

When additionally estimating spatially independent group effects, the user needs to specify xgroup in the resf function by one or more group variables, as follows:

```
x       <- boston.c[,c("CRIM","ZN","INDUS", "CHAS", "NOX","RM", "AGE")]
res   <- resf(y = y, x = x, meig = meig_g, xgroup = xgroup)
res
```

```
## Call:
## resf(y = y, x = x, xgroup = xgroup, meig = meig_g)
##
## ----Coefficients------------------------------
##                 Estimate         SE    t_value      p_value
## (Intercept)  -0.81545943 3.23135854 -0.2523581 8.008871e-01
## CRIM         -0.04596392 0.02505503 -1.8345188 6.728064e-02
## ZN            0.03285021 0.02313784  1.4197611 1.564153e-01
## INDUS         0.03549188 0.11980486  0.2962474 7.671869e-01
## CHAS         -0.62561231 0.72381491 -0.8643264 3.878995e-01
## NOX         -26.38632673 3.88238119 -6.7964286 3.668488e-11
## RM            6.30273567 0.29409796 21.4307357 0.000000e+00
## AGE          -0.06730232 0.01048068 -6.4215611 3.637544e-10
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
```

```
##                        (Intercept)
## random_SE                 5.074794
## Moran.I/max(Moran.I)      0.812936
##
## Group effects:
##           xgroup
## ramdom_SE 4.4404
##
## ----Error statistics-------------------------
##                        stat
## resid_SE          3.2429178
## adjR2(cond)       0.8740022
## rlogLik       -1465.8457138
## AIC            2955.6914276
## BIC            3006.4098677
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

The estimated independent group effects are extracted as

```
res$b_g[[1]][1:5,]# Estimates in the first 5 groups
```

```
##          Estimate       SE   t_value
## xgroup_0 2.165726 2.061093 1.0507657
## xgroup_1 3.747633 1.783543 2.1012294
## xgroup_2 6.544205 1.659184 3.9442318
## xgroup_3 2.431558 1.431325 1.6988163
## xgroup_4 1.036033 1.181672 0.8767521
```

2.2.4.3 Small area estimation Small area estimation (SAE; Ghosh and Rao, 1994) is a statistical technique estimating parameters for small areas such as districts and municipality. SAE is useful for obtaining reliable small area statistics from noisy data. The resf and resf_vc functions are available for SEA (see as explained in Murakami 2020 for further detail).

The Boston housing datasets consist of 506 samples in 92 towns. This section estimates the standard housing price in the I-th towns by assuming the following model:

$$y_I = \hat{y}_I + \epsilon_I, \qquad \epsilon_I \sim N(0, \frac{\sigma^2}{N_I})$$

where $\hat{y}_I = \sum_{i=1}^{N_I} \frac{\hat{y}_i}{N_I}$. This model decomposes the observed mean house price y_I in the I-th town into the standard price $\hat{y}_I$ and noise ϵ_I, which reduces as the number of samples in the I-th town increases. The standard price is defined by an aggregate of the predictors $\hat{y}_i$ by individuals.

The above equation suggests that, if $\hat{y}_i$ is predicted using the resf or resf_vc function and aggregated into the towns, we can estimate the standard house price. Here is a sample code for the individual level prediction:

```
r_res <-resf(y=y, x=x, meig=meig_g, xgroup=xgroup)
pred  <-predict0(r_res, x0=x, meig0=meig_g, xgroup0=xgroup)
pred$pred[1:5,]
```

```
##       pred       xb sf_residual   xgroup
## 1 23.70932 22.71407   -1.170482 2.165726
## 2 24.57615 22.21874   -1.390220 3.747633
## 3 30.58942 28.23201   -1.390220 3.747633
## 4 33.24998 28.19959   -1.493814 6.544205
```

```
## 5 33.62206 28.57167   -1.493814 6.544205
```

As shown above, the predict0 function returns predicted values (pred), predicted trends (xb), predicted residual spatial components (sf_residuals), and predicted group effects (xgroup). Then, these individual-level variables are aggregated into towns. Here is a code:

```
adat  <- aggregate(data.frame(y, pred$pred),by=list(xgroup),mean)
adat[1:5,]
```

```
##   Group.1        y     pred       xb sf_residual   xgroup
## 1       0 24.00000 23.70932 22.71407   -1.170482 2.165726
## 2       1 28.15000 27.58279 25.22537   -1.390220 3.747633
## 3       2 32.76667 31.89132 26.84093   -1.493814 6.544205
## 4       3 19.42857 19.36679 18.51187   -1.576641 2.431558
## 5       4 16.71364 16.72781 17.10793   -1.416151 1.036033
```

The outputs are the predicted standard price (pred), trend (xb), spatially dependent group effects (sf_residual), and spatially independent group effects (xgroup) by town.

To map the result, spatial polygons for the towns are loaded and combined with our estimates:

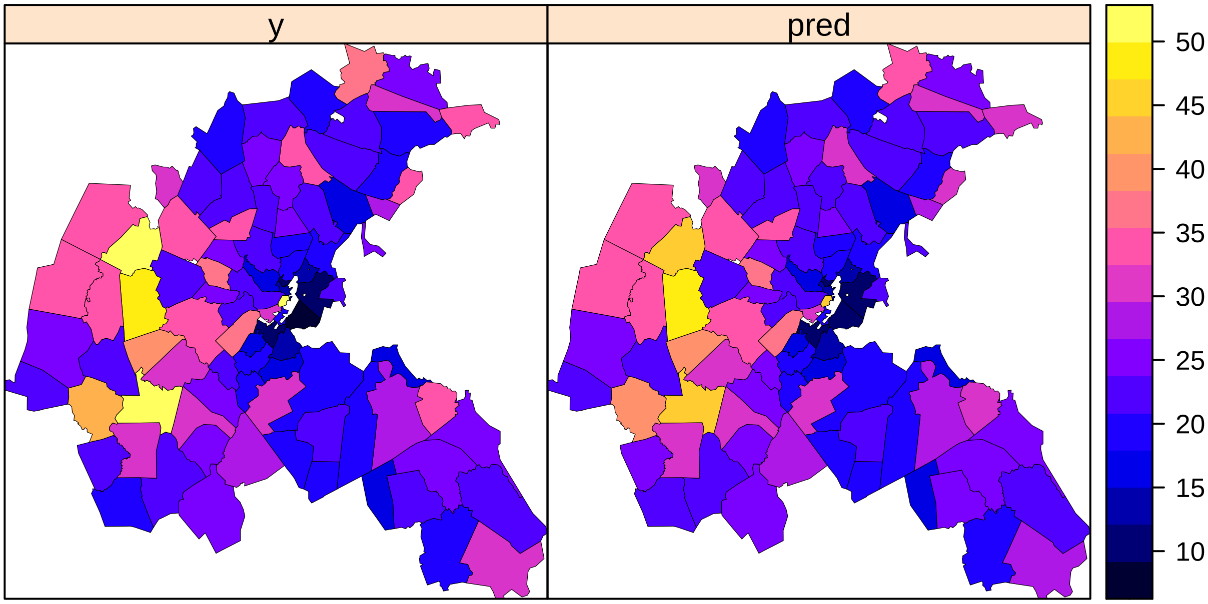
```
require(rgdal)
require(rgeos)
require(dplyr)
boston.tr   <- readOGR(system.file("shapes/boston_tracts.shp",package="spData")[1])
```

```
## OGR data source with driver: ESRI Shapefile
## Source: "/Library/Frameworks/R.framework/Versions/4.0/Resources/library/spData/shapes/boston_tracts.s
## with 506 features
## It has 36 fields
```

```
b1          <- st_as_sf(boston.tr)
b1_dissolve <- b1 %>% group_by(TOWNNO) %>% summarize() #dissolve
boston.tr2  <- as_Spatial(b1_dissolve)
boston.tr2@data$id<- 1:(dim(boston.tr2)[1])
b2_dat      <- boston.tr2@data
b2_dat2     <- merge(b2_dat, adat,by.x="TOWNNO",by.y="Group.1",all.x=TRUE)
```

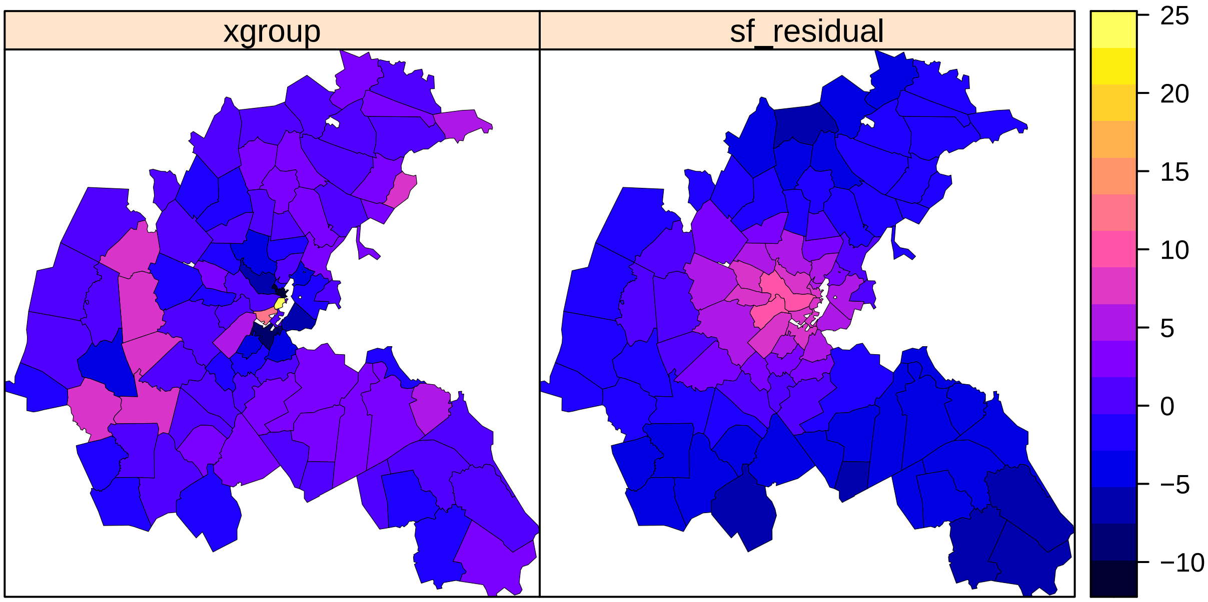
Here are the maps of our estimates. “y” denotes the observed mean prices, and “pred” denotes our predicted standard price. While they are similar, there are some differences in towns with high housing prices.

```
boston.tr2@data<- b2_dat2[order(b2_dat2$id),]
spplot(boston.tr2,c("y","pred"), lwd=0.3)
```

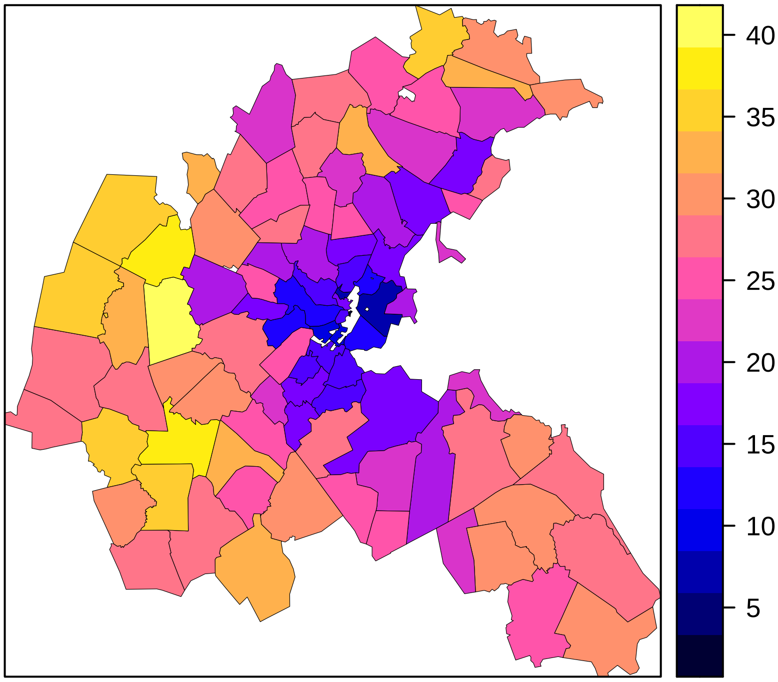


Here are the elements of the predicted values. The maps below show that each element explains different things to each other:

```
spplot(boston.tr2,c("xgroup","sf_residual"), lwd=0.3)
```




```
spplot(boston.tr2,"xb", lwd=0.3)
```



Note that the resf_vc function is also available for SVC model-based SAE. Here is a sample code:

```
rv_res  <- resf_vc(y=y, x=x, meig=meig_g, xgroup=xgroup, x_sel=FALSE)
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/9"
## [1] "2/9"
## [1] "3/9"
## [1] "4/9"
## [1] "5/9"
## [1] "6/9"
## [1] "7/9"
## [1] "8/9"
## [1] "9/9"
## [1] "BIC: 3074.297"
## [1] "-------  Iteration 2  -------"
## [1] "1/9"
## [1] "2/9"
## [1] "3/9"
## [1] "4/9"
## [1] "5/9"
## [1] "6/9"
## [1] "7/9"
## [1] "8/9"
## [1] "9/9"
## [1] "BIC: 3040.896"
## [1] "-------  Iteration 3  -------"
## [1] "1/9"
## [1] "2/9"
## [1] "3/9"
## [1] "4/9"
## [1] "5/9"
## [1] "6/9"
```

```
## [1] "7/9"
## [1] "8/9"
## [1] "9/9"
## [1] "BIC: 3039.588"
## [1] "-------  Iteration 4  -------"
## [1] "1/9"
## [1] "2/9"
## [1] "3/9"
## [1] "4/9"
## [1] "5/9"
## [1] "6/9"
## [1] "7/9"
## [1] "8/9"
## [1] "9/9"
## [1] "BIC: 3039.572"
## [1] "-------  Iteration 5  -------"
## [1] "1/9"
## [1] "2/9"
## [1] "3/9"
## [1] "4/9"
## [1] "5/9"
## [1] "6/9"
## [1] "7/9"
## [1] "8/9"
## [1] "9/9"
## [1] "BIC: 3039.572"
```

```
pred_vc <- predict0_vc(rv_res, x0=x, meig0=meig_g, xgroup0=xgroup)
adat_vc <- aggregate(data.frame(y, pred_vc$pred), by=list(xgroup), mean)
adat_vc[1:5,]
```

```
##   Group.1        y     pred       xb sf_residual   xgroup
## 1       0 24.00000 23.67839 23.12533   -1.125536 1.678592
## 2       1 28.15000 27.81181 27.44629   -1.966846 2.332368
## 3       2 32.76667 32.28629 31.09675   -2.552106 3.741645
## 4       3 19.42857 19.25653 18.45742   -2.506070 3.305184
## 5       4 16.71364 16.68358 15.40519   -1.025996 2.304387
```

2.2.4.4 Longitudinal/panel data analysis The resf and resf_vc functions are also available for longitudinal or panel data analysis with/without S(N)VC (see Yu et al., 2020). Although this section takes resf as an example, resf_vc function-based panel analysis is implemented in the same way.

To illustrate this, we use a panel data of 48 US states from 1970 to 1986, which is published in the plm package (Croissant and Millo, 2008). Because our approach uses spatial coordinates by default, we added center spatial coordinates (px and py) to the panel data. Here is the code:

```
require(plm)
require(spData)

data(Produc, package = "plm")
data(us_states)
us_states2 <- data.frame(us_states$GEOID,us_states$NAME,st_coordinates(st_centroid(us_states)))
names(us_states2)[3:4]<- c("px","py")
us_states3 <- us_states2[order(us_states2[,1]),][-8,]
us_states3$state<- unique(Produc[,1])
```

```r
pdat0      <- na.omit(merge(Produc,us_states3[,c(3:5)],by="state",all.x=TRUE,sort=FALSE))
pdat       <- pdat0[order(pdat0$state,pdat0$year),]
pdat[1:5,]
```

```
##     state year region     pcap     hwy   water    util       pc   gsp    emp
## 1 ALABAMA 1970      6 15032.67 7325.80 1655.68 6051.20 35793.80 28418 1010.5
## 2 ALABAMA 1971      6 15501.94 7525.94 1721.02 6254.98 37299.91 29375 1021.9
## 3 ALABAMA 1972      6 15972.41 7765.42 1764.75 6442.23 38670.30 31303 1072.3
## 4 ALABAMA 1973      6 16406.26 7907.66 1742.41 6756.19 40084.01 33430 1135.5
## 5 ALABAMA 1974      6 16762.67 8025.52 1734.85 7002.29 42057.31 33749 1169.8
##   unemp        px      py
## 1   4.7 -86.82645 32.7926
## 2   5.2 -86.82645 32.7926
## 3   4.7 -86.82645 32.7926
## 4   3.9 -86.82645 32.7926
## 5   5.5 -86.82645 32.7926
```

Here are the first five rows of the data:

```r
pdat[1:5,]
```

```
##     state year region     pcap     hwy   water    util       pc   gsp    emp
## 1 ALABAMA 1970      6 15032.67 7325.80 1655.68 6051.20 35793.80 28418 1010.5
## 2 ALABAMA 1971      6 15501.94 7525.94 1721.02 6254.98 37299.91 29375 1021.9
## 3 ALABAMA 1972      6 15972.41 7765.42 1764.75 6442.23 38670.30 31303 1072.3
## 4 ALABAMA 1973      6 16406.26 7907.66 1742.41 6756.19 40084.01 33430 1135.5
## 5 ALABAMA 1974      6 16762.67 8025.52 1734.85 7002.29 42057.31 33749 1169.8
##   unemp        px      py
## 1   4.7 -86.82645 32.7926
## 2   5.2 -86.82645 32.7926
## 3   4.7 -86.82645 32.7926
## 4   3.9 -86.82645 32.7926
## 5   5.5 -86.82645 32.7926
```

Following a vignette of the plm package, this section uses logged gross state product as explained variables (y) and logged public capital stock (log_pcap), logged private capital stock (log_pc), logged labor input measured by the employment in non-agricultural payrolls (log_emp), and unemployment rate (unemp) as covariables.

```r
y      <- log(pdat$gsp)
x      <- data.frame(log_pcap=log(pdat$pcap), log_pc=log(pdat$pc),
                     log_emp=log(pdat$emp), unemp=pdat$unemp)
```

Because spatial coordinates are defined by states, Moran eigenvectors must be extracted by state by specifying s_id in the meigen function, as follows:

```r
coords<- pdat[,c("px", "py")]
s_id  <- pdat$state
meig_p<- meigen(coords,s_id=s_id)# Moran eigenvectors by states
```

Currently, the following spatial panel models are available: pooling model (no group effects); individual random effects model (state-level group effects); time random effects model (year-level group effects); and two-way random effects model (state and year-level group effects). All these models consider residual spatial dependence. Here are the codes implementing these models:

```r
pmod0 <- resf(y=y,x=x,meig=meig_p) # pooling model
```

```r
xgroup<- pdat[,c("state")]
pmod1 <- resf(y=y,x=x,meig=meig_p,xgroup=xgroup)# individual model

xgroup<- pdat[,c("year")]
pmod2 <- resf(y=y,x=x,meig=meig_p,xgroup=xgroup)# time model

xgroup<- pdat[,c("state","year")]
pmod3 <- resf(y=y,x=x,meig=meig_p,xgroup=xgroup)# two-way model
```

Among these models, the two-way model indicates the smallest BIC. The output is summarized as

```r
pmod3
```

```
## Call:
## resf(y = y, x = x, xgroup = xgroup, meig = meig_p)
##
## ----Coefficients------------------------------
##                 Estimate          SE    t_value      p_value
## (Intercept)  2.266458952 0.157678635 14.3739128 0.0000000000
## log_pcap     0.007185026 0.023530593  0.3053483 0.7601856016
## log_pc       0.292350481 0.022207172 13.1646874 0.0000000000
## log_emp      0.732900408 0.024808722 29.5420464 0.0000000000
## unemp       -0.004356469 0.001066686 -4.0841178 0.0000490012
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
##                       (Intercept)
## random_SE               0.1555241
## Moran.I/max(Moran.I)    0.3344001
##
## Group effects:
##                state       year
## ramdom_SE 0.09492895 0.02433059
##
## ----Error statistics--------------------------
##                       stat
## resid_SE     3.381428e-02
## adjR2(cond)  9.988953e-01
## rlogLik      1.408381e+03
## AIC         -2.796762e+03
## BIC         -2.749718e+03
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

The estimated group effects are displayed as follows:

```r
s_g   <- pmod3$b_g[[1]]
s_g[1:5,]# State-level group effects
```

```
##                      Estimate         SE   t_value
## state_ALABAMA    -0.07165160 0.01390024 -5.154702
## state_ARIZONA    -0.04404058 0.01667988 -2.640342
## state_ARKANSAS   -0.07256766 0.01471017 -4.933162
## state_CALIFORNIA  0.24012817 0.01967478 12.204875
```

```
## state_COLORADO   -0.11492607 0.01232067 -9.327905
```

```r
t_g   <- pmod3$b_g[[2]]
t_g[1:5,]# Year-level group effects
```

```
##                Estimate         SE    t_value
## year_1970 -0.006016459 0.01109130 -0.5424484
## year_1971  0.002901673 0.01056932  0.2745372
## year_1972  0.013281801 0.01041698  1.2750149
## year_1973  0.021949386 0.01028021  2.1351098
## year_1974 -0.009852614 0.00967949 -1.0178857
```

For validation, the same panel model (but without spatial dependence) is estimated using the plm function:

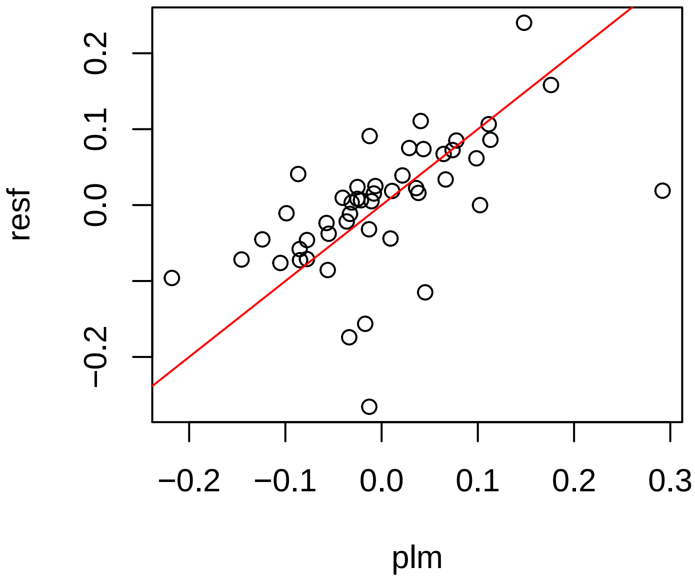
```r
pm0    <- plm(log(gsp) ~ log(pcap) + log(pc) + log(emp) + unemp,
              data = pdat, effect="twoways",model="random")
pm0
```

```
##
## Model Formula: log(gsp) ~ log(pcap) + log(pc) + log(emp) + unemp
##
## Coefficients:
## (Intercept)   log(pcap)     log(pc)    log(emp)       unemp
##   2.3634993   0.0178529   0.2655895   0.7448989  -0.0045755
```

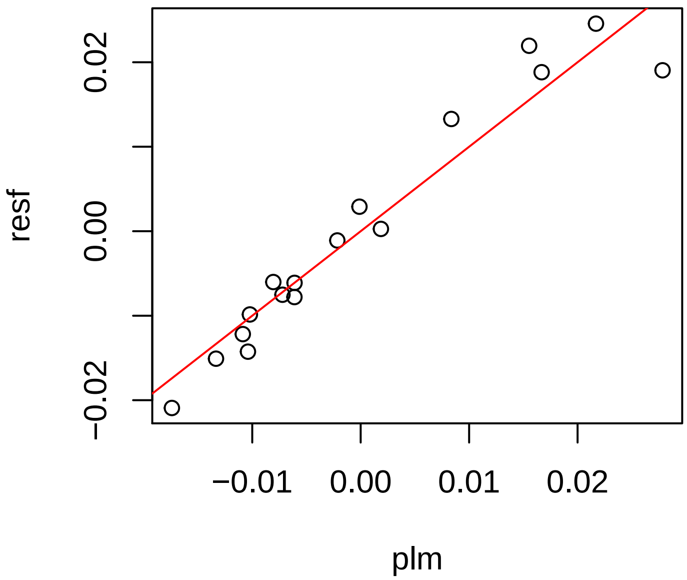
```r
s_g_plm<- ranef(pm0,"individual")
t_g_plm<- ranef(pm0,"time")
```

The coefficient estimates are similar. The plots below compare estimated group effects. Estimated state-level effects have differences because our models consider residual spatial dependence, while plm does not (by default). Time effects are quite similar.

```r
plot(s_g_plm,s_g[,1],xlab="plm",ylab="resf")
abline(0,1,col="red")
```




```r
plot(t_g_plm,t_g[,1],xlab="plm",ylab="resf")
abline(0,1,col="red")
```



2.3 Spatial prediction

This package provides functions for ESF/RE-ESF-based spatial interpolation minimizing the expected prediction error (just like kriging). RE-ESF approximates a Gaussian process or the kriging model, which has actively been used for spatial prediction, and ESF is a special case (Murakami and Griffith, 2015). Because ESF and RE-ESF perform approximations, their spatial predictions might be less accurate relative to kriging. Instead, they are faster and available for very large samples.

The predict0 function is used for prediction based on the resf or besf function, while the predict0_vc function is used for resf_vc or besf_vc function (see Section 4 for besf and besf_vc functions).

In this tutorial, the Lucas housing price data with sample size being 25,357 is used. In the prediction, “price” is used as the explained variable, and “age,” “rooms,” “beds,” and “year” are used as covariates.

```
require(spData)
data(house)
dat   <- data.frame(coordinates(house), house@data[,c("price","age","rooms","beds","syear")])
```

A total of 20,000 randomly selected samples are used for model estimation, and the other 5,357 samples are used for accuracy evaluation. The code below creates the data for observation sites (coords, y, x) and for unobserved sites (coords0, y0, x0):

```
samp    <- sample(dim(dat)[1], 20000)
coords<- dat[samp ,c("long","lat")]
y       <- log(dat[samp,"price"])
x     <- dat[samp,c("age","rooms","beds","syear")]

coords0<- dat[-samp ,c("long","lat")]
y0    <- log(dat[-samp,"price"]) # for valudation
x0    <- dat[-samp,c("age","rooms","beds","syear")]
```

The prediction is done in two steps: (1) evaluation of Moran eigenvectors at prediction sites using the meigen0 function; (2) prediction using the predict0 function. Below is a sample code based on the resf function:

```
start.time1<-proc.time()###### just for CP time evaluation
meig     <- meigen_f(coords)
meig0    <- meigen0( meig=meig, coords0=coords0 )
```

```r
mod       <- resf( y = y, x = x, meig = meig )
pred0      <- predict0( mod = mod, x0 = x0, meig0=meig0 )
end.time1<- proc.time()###### just for CP time evaluation
```

Note that the first and last lines are just for computing time evaluation. NVCs are considered if adding NVC=TRUE in the resf function. The meigen_f function is used for fast computation.

The outputs shown below include predicted values (pred), predicted trend (xb), and predicted residual spatial component (sf_residuals).

```r
pred0$pred[1:5,]
```

```
##        pred       xb sf_residual
## 4  11.26823 10.77544   0.4927898
## 5  11.89102 11.33839   0.5526369
## 6  11.52725 11.03301   0.4942443
## 13 12.28207 11.70984   0.5722306
## 24 11.71981 11.22721   0.4926028
```

```r
pred     <- pred0$pred[,1]
```

On the other hand, here is a code for a spatial prediction based on an S(N)VC model:

```r
start.time2<-proc.time()###### just for CP time evaluation
meig     <- meigen_f(coords)
meig0    <- meigen0( meig=meig, coords0=coords0 )
mod2       <- resf_vc( y = y, x = x, meig = meig )
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13521.406"
## [1] "-------  Iteration 2  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13130.608"
## [1] "-------  Iteration 3  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13126.759"
## [1] "-------  Iteration 4  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13126.645"
```

```
## [1] "-------  Iteration 5  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13126.643"
## [1] "-------  Iteration 6  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 13126.643"
```

```
pred02   <- predict0_vc( mod = mod2, x0 = x0, meig0=meig0 )
end.time2<- proc.time()###### just for CP time evaluation
```

NVCs are considered by adding NVC=TRUE in the resf_vc function. Here are the output variables:

```
pred02$pred[1:5,]
```

```
##        pred       xb sf_residual
## 4  11.39284 11.22956   0.1632782
## 5  11.89070 11.74495   0.1457541
## 6  11.48418 11.31117   0.1730038
## 13 12.31150 12.15499   0.1565022
## 24 11.66577 11.46887   0.1969049
```

```
pred2    <- pred02$pred[,1]
```

The root mean squared prediction error (RMSPE) and the computational time of the spatial regression model (resf) are as follows:

```
sqrt(sum((pred-y0)^2)/length(y0))#rmse
```

```
## [1] 0.3543658
```

```
(end.time1 - start.time1)[3]#computational time (second)
```

```
## elapsed
##   8.482
```

while those of the SVC model (resf_vc) are as follows:

```
sqrt(sum((pred2-y0)^2)/length(y0))#rmse
```

```
## [1] 0.3345041
```

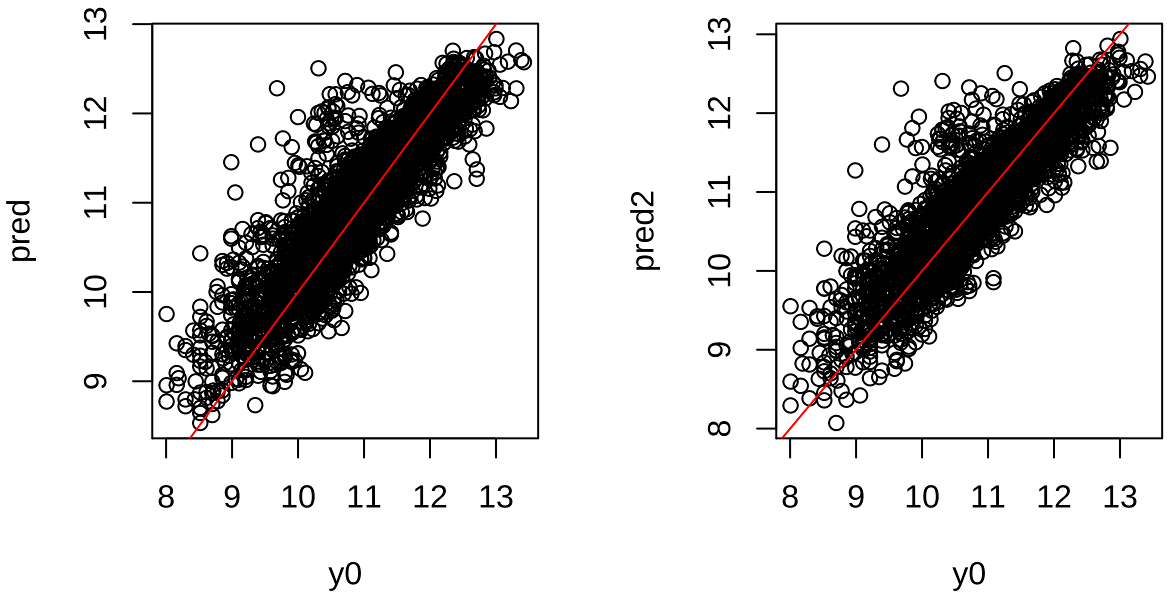
```
(end.time2 - start.time2)[3]#computational time (second)
```

```
## elapsed
##  97.197
```

The results suggest that both models are available for large samples. It is also demonstrated that while the spatial regression model is faster than the SVC model, the SVC model is slightly more accurate. The actual values (y0) and predicted values (pred/pred2) are compared below:

```
par(mfrow=c(1,2))
plot(y0,pred);abline(0,1,col="red")
```

```r
plot(y0,pred2);abline(0,1,col="red")
```



3 Compositionally-warped additive mixed models (CAMM) for non-Gaussian data

3.1 Basic models

Although the previous section applies Gaussian linear models, y does not necessarily follow a Gaussian distribution. For non-Gaussian continuous data, Murakami et al. (2021) proposed the compositionally warped additive mixed model (CAMM) including the following spatial regression model as a special case:

$$\phi_\theta(y_i) = \sum_{k=1}^{K} x_{i,k}\beta_k + f_{MC}(s_i) + \epsilon_i, \qquad \epsilon_i \sim N(0, \sigma^2),$$

$\phi_\theta(y_i)$ is defined by concatenating D transformation functions as follows:

$$\phi_\theta(y_i) = \phi_{\theta_D}(\ldots(\phi_{\theta_2}(\phi_{\theta_1}(y_i))\ldots),$$

where $\theta \in \{\theta_1, \ldots, \theta_D\}$. The d-th transformation function is defined as

$$\phi_{\theta_d}(y_i) = \theta_{d,1} + \theta_{d,2}\sinh\{\theta_{d,3}\text{arcsinh}(y) + \theta_{d,4}\}$$

$\theta_d \in \{\theta_{d,1}, \theta_{d,2}, \theta_{d,3}, \theta_{d,4}\}$ are parameters. Based on Rois and Tober (2019), the transformation $\phi_\theta(y_i)$, which they called the SAL transformation, approximates a wide variety of non-Gaussian continuous distributions without explicitly assuming data distribution. Therefore, this approach is available for a wide range of non-Gaussian continuous data (See Figure 1). Figure 2 illustrates the transformation function $\phi_\theta(y_i)$ implemented in this package. Below, we explain the transformations (A) and (B).

The transformation approach is readily estimated by specifying the number of transformations tr_num (=D). For example, the model with one SAL transformation is estimated as follows:

```r
y          <- boston.c[, "CMEDV"]
x        <- boston.c[,c("CRIM", "AGE")]
```

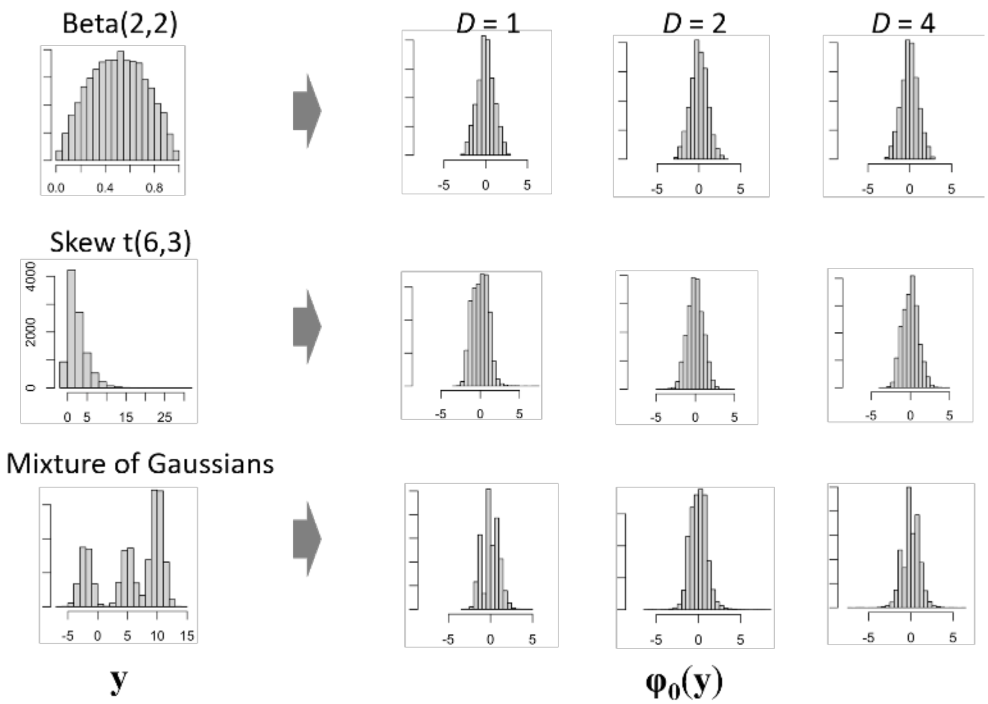


Figure 1: Test resuts of the transfromation approach of Rois and Tober (2019). Left three panels represent histograms of the simulated data generated from beta distribution, skew t distribution, and Gaussian mixtures respectively. The right nine panels show the histograms after the transformation. D is the number of transformations. This figure confirms that this approach accurately transformes a wide variety of non-Gaussian distributions to Gaussian distributions

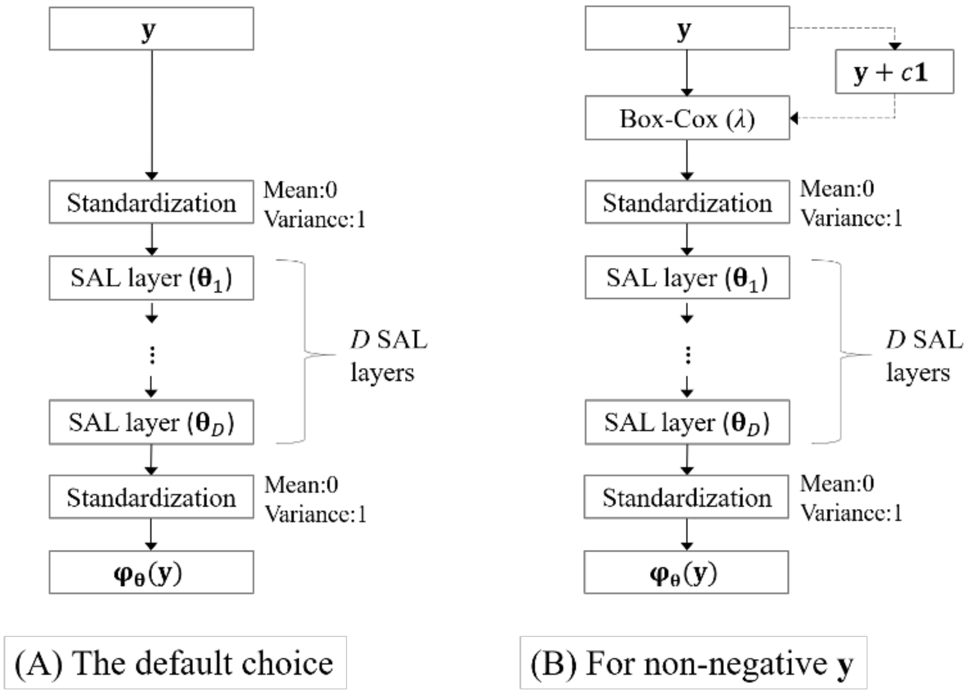


Figure 2: Transformation functions assumed in this package. (A) is the default function while (B) is recommended for non-negative y.

```
xconst  <- boston.c[,c("ZN","DIS","RAD","NOX",  "TAX","RM", "PTRATIO", "B")]
coords  <- boston.c[,c("LON","LAT")]
meig      <- meigen(coords=coords)
res_c1  <- resf(y = y, x = x, meig = meig, tr_num=1)
res_c1
```

```
## Call:
## resf(y = y, x = x, meig = meig, tr_num = 1)
##
## ----Coefficients------------------------------
##                Estimate          SE   t_value      p_value
## (Intercept)  0.80897890 0.124073447  6.520161 1.859539e-10
## CRIM        -0.03065298 0.003712140 -8.257494 1.554312e-15
## AGE         -0.01018177 0.001773668 -5.740514 1.720394e-08
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
##                      (Intercept)
## random_SE              0.6752488
## Moran.I/max(Moran.I)   0.4492342
##
## ----Error statistics--------------------------
##                      stat
## resid_SE        0.5218113
## adjR2(cond)     0.7238702
## rlogLik     -1561.5463361
```

```
## AIC          3141.0926721
## BIC          3179.1315022
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

If tr_num is specified, the explained variables are standardized to mean zero and variance one. The transformed y is returned as

```
res_c1$tr_y[1:10]
```

```
##  [1]  0.4822911  0.1810262  1.2410173  1.1801325  1.3058208  0.9052097
##  [7]  0.3520385  0.2489815 -0.6474152 -0.2300006
```

Because of the standardization, the regression coefficients have different scales from the no transformation models with tr_num = 0 (note: the coefficient values are comparable if y is standardized a priori). Still, the AIC and BIC values, which are defined for y before the transformation, are comparable irrespective of with or without transformations.

The number of transformations can be optimized by minimizing the BIC value. For instance, the model with tr_num = 1 is better than the model with tr_num = 5, which is estimated below, because of the smaller BIC value:

```
res_c5    <- resf(y = y, x = x, meig = meig, tr_num=5)
```

While the regression coefficients quantify the impact on the transformed y, the marginal effects $\partial y_i / \partial x_{i,k}$, which quantify the influence of x on y before the transformation can be evaluated using the coef_marginal function.

```
coef_marginal(res_c1)
```

```
## Call:
## coef_marginal(mod = res_c1)
##
## ----Marginal effects from x (dx_i/dy_i) (summary)-------
##  (Intercept)         CRIM               AGE
##  Mode:logical   Min.   :-1.0920   Min.   :-0.36271
##  NA's:506       1st Qu.:-0.3005   1st Qu.:-0.09982
##                 Median :-0.2146   Median :-0.07128
##                 Mean   :-0.3136   Mean   :-0.10417
##                 3rd Qu.:-0.1715   3rd Qu.:-0.05696
##                 Max.   :-0.1459   Max.   :-0.04846
##
##  Note: Medians are recommended summary statistics
```

The medians might especially be useful as summary statistics.

For non-negative variables, a Box-Cox transformation can be introduced as the first transformation ϕ_{θ_1} by specifying tr_nonneg = TRUE. For example, a model with a Box-Cox transformation and one SAL transformation is implemented as follows:

```
res_c1b   <- resf(y = y, x = x, meig = meig, tr_num=1, tr_nonneg=TRUE)
res_c1b
```

```
## Call:
## resf(y = y, x = x, meig = meig, tr_num = 1, tr_nonneg = TRUE)
##
## ----Coefficients------------------------------
##                 Estimate          SE   t_value      p_value
```

```
## (Intercept)  0.83846115 0.127225089  6.590376 1.208931e-10
## CRIM        -0.02908981 0.003818076 -7.618972 1.481038e-13
## AGE         -0.01069406 0.001818698 -5.880065 7.913461e-09
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
##                      (Intercept)
## random_SE              0.6810151
## Moran.I/max(Moran.I)   0.4432883
##
## ----Error statistics-------------------------
##                      stat
## resid_SE        0.5372400
## adjR2(cond)     0.7072998
## rlogLik     -1561.1862921
## AIC          3140.3725842
## BIC          3178.4114142
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

The estimated parameter for the Box-Cox transfromation is dispalyed as

```
res_c1b$tr_bpar
```

```
##                                             Estimates
## Lambda (Box-Cox)                              1.78342
## Constant added on y (nonzero if y has zero)   0.00000
```

where "Lambda (Box-Cox)" is the lambda parameter for the transformation. lambda = 0 means log-transformation while lambda = 1 means no transformation. Because the Box-Cox transformation is available only for positive values, another parameter c (>0) is estimated during the REML (or ML) and the transformation is employed to $y + c$, which is always positive. "Constant added on y (nonzero if y has zero)" is the c parameter that has zero value when y does not have zero values like our case.

Based on the BIC value, the model with the Box-Cox and single SAL transformation (res_c1b) has the highest accuracy. If tr_nonneg = TRUE and y has zero values, a small value is added on y to make the Box-Cox transformation feasible. The small value is estimated during the estimation procedure. See Murakami et al. (2021) for further detail.

3.2 Extended models

The resf_vc function, which was used for Gaussian regression modeling, is also available for compositionally warped additive mixed modeling (CAMM). As explained above, CAMM models a wide variety of non-Gaussian continuous data without explicit assumption on data distribution. When modeling SNVCs, the CAMM is defined as follows:

$$\phi_\theta(y_i) = \sum_{k=1}^{K} x_{i,k}\beta_{i,k} + f_{MC}(s_i) + \epsilon_i, \qquad \beta_{i,k} = b_k + f_{MC,k}(s_i) + f(x_{i,k}), \qquad \epsilon_i \sim N(0, \sigma^2),$$

where $\phi_\theta(y_i) = \phi_{\theta_D}(\ldots(\phi_{\theta_2}(\phi_{\theta_1}(y_i))\ldots)$ concatenates D transformation functions as illustrated in Figure 2.

The CAMM with SNVCs is estimated by specifying the number of transformations tr_num (=D). For example, the model with one SAL transformation is estimated as follows:

```r
res_c1    <- resf_vc( y=y,x=x,xconst=xconst,meig=meig, x_nvc=TRUE, tr_num=1)
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2908.683"
## [1] "-------  Iteration 2  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2907.435"
## [1] "-------  Iteration 3  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2907.433"
## [1] "-------  Iteration 4  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2907.433"
```

```r
res_c1
```

```
## Call:
## resf_vc(y = y, x = x, xconst = xconst, x_nvc = TRUE, meig = meig,
##     tr_num = 1)
##
## ----Spatially and non-spatially varying coefficients on x (summary)----
##
## Coefficient estimates:
##   (Intercept)              CRIM                 AGE
##  Min.   :-0.0006279   Min.   :-0.2633293   Min.   :-0.018085
##  1st Qu.:-0.0006279   1st Qu.:-0.0601699   1st Qu.:-0.010344
##  Median :-0.0006279   Median :-0.0314600   Median :-0.007331
##  Mean   :-0.0006279   Mean   :-0.0334682   Mean   :-0.007189
##  3rd Qu.:-0.0006279   3rd Qu.:-0.0004735   3rd Qu.:-0.003945
##  Max.   :-0.0006279   Max.   : 0.0984538   Max.   : 0.004902
##
## Statistical significance:
##                            Intercept CRIM AGE
## Not significant                  506  413 122
## Significant (10% level)            0   13  22
## Significant ( 5% level)            0   21  53
## Significant ( 1% level)            0   59 309
```

```
##
## ----Constant coefficients on xconst----------------------------
##              Estimate           SE   t_value      p_value
## ZN       0.002131141 0.0011581112  1.840187 6.642196e-02
## DIS     -0.125853881 0.0235736802 -5.338746 1.510766e-07
## RAD      0.052505481 0.0085076651  6.171550 1.550439e-09
## NOX     -3.068957591 0.4535176701 -6.767008 4.263945e-11
## TAX     -0.001673300 0.0003116946 -5.368398 1.295405e-07
## RM       0.492531195 0.0294958844 16.698302 0.000000e+00
## PTRATIO -0.055585085 0.0134669668 -4.127513 4.396602e-05
## B        0.002461353 0.0002838022  8.672773 0.000000e+00
##
## ----Variance parameters----------------------------------
##
## Spatial effects (coefficients on x):
##                       (Intercept)       CRIM        AGE
## random_SE            4.027931e-06 0.12655766 0.00685466
## Moran.I/max(Moran.I) 4.999344e-01 0.05050829 0.28484937
##
## Non-spatial effects (coefficients on x):
##                  CRIM AGE
## random_SE 0.003847197   0
##
## ----Error statistics-------------------------------------
##                        stat
## resid_SE          0.3292694
## adjR2(cond)       0.8888881
## rlogLik       -1385.2247972
## AIC            2814.4495943
## BIC            2907.4334010
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x and xconst)
```

As is the case in the resf_function, the explained variables are standardized to mean zero and variance one. Because of the standardization, the estimated coefficients have different scales from the no transformation models with tr_num = 0. Still, the AIC and BIC values are comparable irrespective of with or without transformation. The number of transformation (tr_num) can be optimized by minimizing the BIC value as illustrated in the previous section.

For non-negative y, a Box-Cox transformation can be introduced as the first transformation ϕ_{θ_1} by specifying tr_nonneg = TRUE (see Figure 2 (B)). For example, a model with a Box-Cox transformation and one SAL transformation is implemented as follows:

```
res_c1b   <- resf_vc(y=y,x=x,xconst=xconst,meig=meig, x_nvc=TRUE, tr_num=1, tr_nonneg=TRUE)
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2905.576"
## [1] "-------  Iteration 2  -------"
## [1] "1/5"
## [1] "2/5"
```

```
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2904.38"
## [1] "-------  Iteration 3  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2901.644"
## [1] "-------  Iteration 4  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2901.641"
## [1] "-------  Iteration 5  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2901.641"
## [1] "-------  Iteration 6  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 2901.641"
```

```
res_c1b
```

```
## Call:
## resf_vc(y = y, x = x, xconst = xconst, x_nvc = TRUE, meig = meig,
##     tr_num = 1, tr_nonneg = TRUE)
##
## ----Spatially and non-spatially varying coefficients on x (summary)----
##
## Coefficient estimates:
##   (Intercept)            CRIM                  AGE
##  Min.   :-0.02247   Min.   :-0.2742262   Min.   :-0.018916
##  1st Qu.:-0.02247   1st Qu.:-0.0599819   1st Qu.:-0.010590
##  Median :-0.02247   Median :-0.0322698   Median :-0.007598
##  Mean   :-0.02247   Mean   :-0.0329547   Mean   :-0.007425
##  3rd Qu.:-0.02247   3rd Qu.: 0.0004868   3rd Qu.:-0.004352
##  Max.   :-0.02247   Max.   : 0.1072723   Max.   : 0.005453
##
## Statistical significance:
##                          Intercept CRIM AGE
## Not significant                506  410 117
## Significant (10% level)          0   19  24
## Significant ( 5% level)          0   18  52
```

```
## Significant ( 1% level)         0   59 313
##
## ----Constant coefficients on xconst----------------------------
##             Estimate           SE   t_value      p_value
## ZN       0.002027189 0.0011644956  1.740830 8.242154e-02
## DIS     -0.131273177 0.0237815167 -5.519967 5.842444e-08
## RAD      0.052234302 0.0085594228  6.102550 2.314237e-09
## NOX     -3.124666387 0.4565361590 -6.844291 2.628964e-11
## TAX     -0.001635804 0.0003135690 -5.216727 2.825564e-07
## RM       0.507013270 0.0296304870 17.111203 0.000000e+00
## PTRATIO -0.056302657 0.0135698124 -4.149111 4.017032e-05
## B        0.002452444 0.0002849715  8.605926 0.000000e+00
##
## ----Variance parameters----------------------------------
##
## Spatial effects (coefficients on x):
##                        (Intercept)       CRIM        AGE
## random_SE            7.675951e-06 0.12909777 0.00702793
## Moran.I/max(Moran.I) 5.448140e-01 0.05167624 0.27379960
##
## Non-spatial effects (coefficients on x):
##                   CRIM AGE
## random_SE 0.003807952   0
##
## ----Error statistics-------------------------------------
##                      stat
## resid_SE        0.3303195
## adjR2(cond)     0.8881782
## rlogLik     -1382.3284169
## AIC          2808.6568338
## BIC          2901.6406406
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x and xconst)
```

The estimated parameter for the Box-Cox transfromation is dispalyed as

```
res_c1b$tr_bpar
```

```
##                                              Estimates
## Lambda (Box-Cox)                              1.691544
## Constant added on y (nonzero if y has zero)  0.000000
```

where "Lambda (Box-Cox)" and "Constant added on y (nonzero if y has zero)" are as explained in the previous section.

The models are estimated while varying tr_num∈{0,1,2,3,4,5} and tr_nonneg∈{TRUE,FALSE}, the BIC takes the minimim value when tr_num=1 and tr_nonneg=TRUE. In other words, the model with the Box-Cox and single SAL transformation (res_c1b) has the highest accuracy.

The marginal effects $\partial y_i / \partial x_{i,k}$ quantifying the influence of x and xconst on y before the transformation can be evaluated using the coef_marginal_vc function as follows:

```
coef_marginal_vc(res_c1)
```

```
## Call:
## coef_marginal_vc(mod = res_c1)
```

```
##
## ----Marginal effects from x (dx_i/dy_i) (summary)----
##  (Intercept)         CRIM               AGE
##  Mode:logical   Min.   :-3.019846   Min.   :-0.28655
##  NA's:506       1st Qu.:-0.485275   1st Qu.:-0.07745
##                 Median :-0.236015   Median :-0.05772
##                 Mean   :-0.296976   Mean   :-0.05758
##                 3rd Qu.:-0.006119   3rd Qu.:-0.03649
##                 Max.   : 1.285607   Max.   : 0.14566
##
## ----Marginal effects from xconst (dx_i/dy_i)(summary)----
##        ZN               DIS               RAD              NOX
##  Min.   :0.01173   Min.   :-3.7395   Min.   :0.2891   Min.   :-91.19
##  1st Qu.:0.01254   1st Qu.:-1.1514   1st Qu.:0.3090   1st Qu.:-28.08
##  Median :0.01480   Median :-0.8742   Median :0.3647   Median :-21.32
##  Mean   :0.02026   Mean   :-1.1965   Mean   :0.4992   Mean   :-29.18
##  3rd Qu.:0.01950   3rd Qu.:-0.7408   3rd Qu.:0.4804   3rd Qu.:-18.06
##  Max.   :0.06332   Max.   :-0.6929   Max.   :1.5601   Max.   :-16.90
##       TAX                  RM            PTRATIO              B
##  Min.   :-0.049719   Min.   : 2.712   Min.   :-1.6516   Min.   :0.01355
##  1st Qu.:-0.015308   1st Qu.: 2.899   1st Qu.:-0.5085   1st Qu.:0.01449
##  Median :-0.011624   Median : 3.421   Median :-0.3861   Median :0.01710
##  Mean   :-0.015908   Mean   : 4.683   Mean   :-0.5285   Mean   :0.02340
##  3rd Qu.:-0.009849   3rd Qu.: 4.506   3rd Qu.:-0.3272   3rd Qu.:0.02252
##  Max.   :-0.009213   Max.   :14.635   Max.   :-0.3060   Max.   :0.07313
##
##  Note: Medians are recommended summary statistics
```

3.3 How to use

For most continuous data, it is reasonable to implement CAMM following the way explained above. Exceptionally, if continuous data have many zero values, it might be reasonable to assign greater weights on larger observations by specifying weight in the resf and resf_vc functions. Here is a sample code of a weighted CAMM:

```r
res_w1  <- resf(y = y, x = x, meig = meig, tr_num=1, tr_nonneg=TRUE, weight = y + 0.5)
```

The weights must be positive. If the weight variables, which is y in this example, have zeros a small constant can be added as illustrated above.

For count data, they can be transformed to continuous data e.g. by dividing area or population a priori. Alternatively, CAMM can be specified to estimate a quasi-Poisson model with unknown link function. Based on Chan and Vasconcelos (2011), $N(log(y+c),(y+c)^{-1})$ approximates an over-dispersed Poisson distribution where c is a positive constant. Given that, the following code estimates a quasi-Poisson spatial regression model with log link function:

```r
res_w2  <- resf(y = log(y), x = x, meig = meig, weight = y)
```

while the following code estimates a quasi-Piosson model with unknown link function:

```r
res_w3  <- resf(y = y, x = x, meig = meig, tr_num=1, tr_nonneg=TRUE, weight = y)
```

The link function is estimated from data using the Box-Cox and SAL transformations.

4 Spatially filtered unconditional quantile regression

While the usual (conditional) quantile regression (CQR) estimates the influence of x_k on the τ-th conditional quantile of y, $q_\tau(y|x_k)$, the unconditional quantile regression estimates the influence of x_k on the "unconditional" quantile of y, $q_\tau(y)$ (Firpo et al., 2009).

Suppose that y and x_k represent land price and accessibility, respectively. UQR estimates the influence of accessibility on land price by quantile; it is interpretable and useful for hedonic land price analysis, for example. By contrast, this interpretation does not hold for CQR because it estimates the influence of accessibility on conditional land prices (land price conditional on explanatory variables). Higher conditional land price does not mean higher land price; rather, it means overprice relative to the price expected by the explanatory variables. Therefore, CQR has difficulty in its interpretation, in some cases, including hedonic land price modeling.

The spatial filter UQR (SF-UQR) model (Murakami and Seya, 2019), which is implemented in this package, is formulated as

$$q_\tau(y_i) = \sum_{k=1}^{K} x_{i,k}\beta_{k,\tau} + f_{MC,\tau}(s_i) + \epsilon_{i,\tau}, \qquad \epsilon_{i,\tau} \sim N(0, \sigma_\tau^2),$$

This model is a UQR considering spatial dependence.

The resf_qr function implements this model. Below is a sample code. If boot=TRUE in resf_qr, a semiparametric bootstrapping is performed to estimate the standard errors of the regression coefficients. By default, this function estimates models at 0.1, 0.2,. . . , 0.9 quantiles.

```
y       <- boston.c[, "CMEDV" ]
x       <- boston.c[,c("CRIM","ZN","INDUS", "CHAS", "NOX","RM", "AGE")]
coords<- boston.c[,c("LON","LAT")]
meig    <- meigen(coords=coords)
res   <- resf_qr(y=y,x=x,meig=meig, boot=TRUE)
```

```
## [1] "------- Complete: tau=0.1 -------"
## [1] "------- Complete: tau=0.2 -------"
## [1] "------- Complete: tau=0.3 -------"
## [1] "------- Complete: tau=0.4 -------"
## [1] "------- Complete: tau=0.5 -------"
## [1] "------- Complete: tau=0.6 -------"
## [1] "------- Complete: tau=0.7 -------"
## [1] "------- Complete: tau=0.8 -------"
## [1] "------- Complete: tau=0.9 -------"
```

Here is a summary of the estimation result:

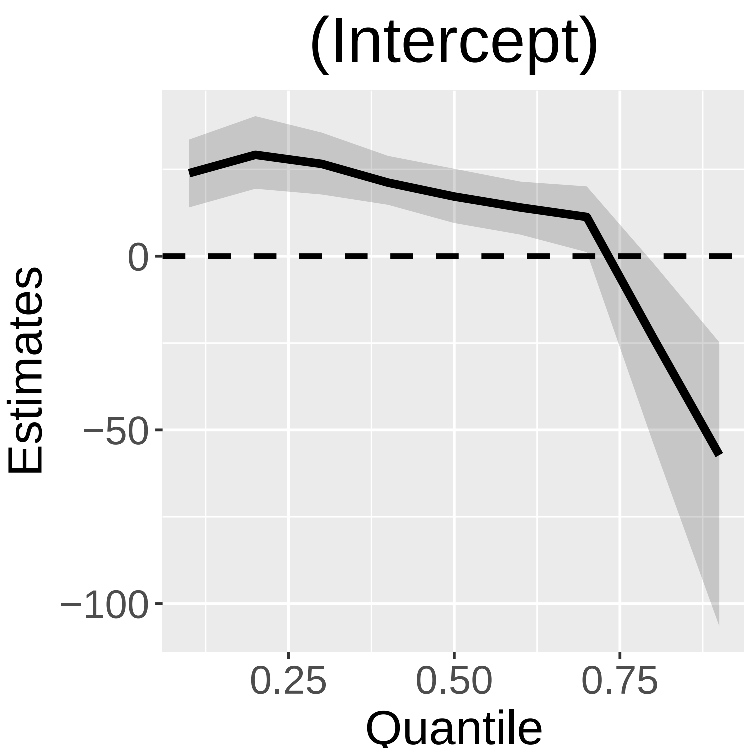
```
res
```

```
## Call:
## resf_qr(y = y, x = x, meig = meig, boot = TRUE)
##
## ----Coefficients------------------------------
##                 tau=0.1      tau=0.2       tau=0.3      tau=0.4       tau=0.5
## (Intercept)  23.86841970  29.16185736  26.550125353  21.16263694  17.151053980
## CRIM         -0.36845124  -0.21172051  -0.106949379  -0.08357496  -0.070290258
## ZN           -0.01169653  -0.01627637  -0.009652286  -0.01947512  -0.008198579
## INDUS         0.25009373   0.03992002  -0.111010420  -0.01521113  -0.096468769
## CHAS          0.98647836   1.28770409   0.438428954   0.26777796  -0.048278485
## NOX         -32.89857783 -23.60303480 -15.109338348 -12.70090129 -11.263158727
## RM            0.71728433   0.49201634   1.169115918   2.21382993   3.004059676
## AGE           0.01977978  -0.05087471  -0.082548477  -0.11192561  -0.105681036
```

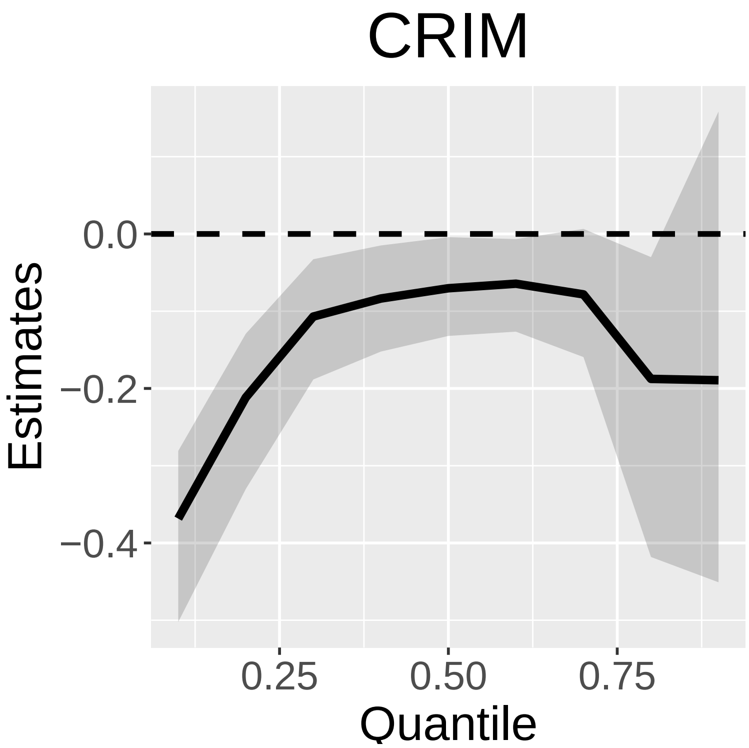
```
##                  tau=0.6      tau=0.7     tau=0.8      tau=0.9
## (Intercept)  13.999671526  11.28433168 -23.3939330 -57.24239068
## CRIM         -0.064412593  -0.07823561  -0.1876252  -0.18934294
## ZN            0.007962903   0.01009742   0.1635369   0.03890142
## INDUS        -0.167039581  -0.30344029  -0.9074079  -0.49797629
## CHAS         -1.665298913  -1.51518801  -3.8773852  -0.04635798
## NOX         -11.405913169 -20.36309658 -39.1980207 -41.26421537
## RM            3.730680883   5.25253569  13.7698457  19.62200618
## AGE          -0.092068861  -0.07567382  -0.0587608  -0.03904752
##
## ----Spatial effects (residuals)---------------
##                                tau=0.1   tau=0.2   tau=0.3   tau=0.4   tau=0.5
## spcomp_SE                    7.1522586 8.1254770 5.7952363 4.4135132 4.7198329
## spcomp_Moran.I/max(Moran.I) 0.2375865 0.3228553 0.3239407 0.3650454 0.5096847
##                                tau=0.6   tau=0.7    tau=0.8    tau=0.9
## spcomp_SE                    4.8818059 6.3633073 16.9989855 16.3826940
## spcomp_Moran.I/max(Moran.I) 0.5690447 0.6935049  0.6757823  0.7203891
##
## ----Error statistics-------------------------
##                      tau=0.1   tau=0.2  tau=0.3   tau=0.4   tau=0.5   tau=0.6
## resid_SE          6.4395412 6.2086846 5.169030 4.7999618 4.5977255 4.8160068
## quasi_adjR2(cond) 0.6007294 0.6828421 0.666506 0.6183801 0.6229795 0.6121279
##                      tau=0.7    tau=0.8    tau=0.9
## resid_SE          5.6288391 12.2961444 18.6716254
## quasi_adjR2(cond) 0.6153019  0.6741455  0.4582676
```

The estimated coefficients can be visualized using the plot_qr function, as below. The numbers 1 to 5 specify which coefficients are plotted (1: intercept). In each panel, solid lines are estimated coefficients, and gray areas are their 95% confidence intervals.

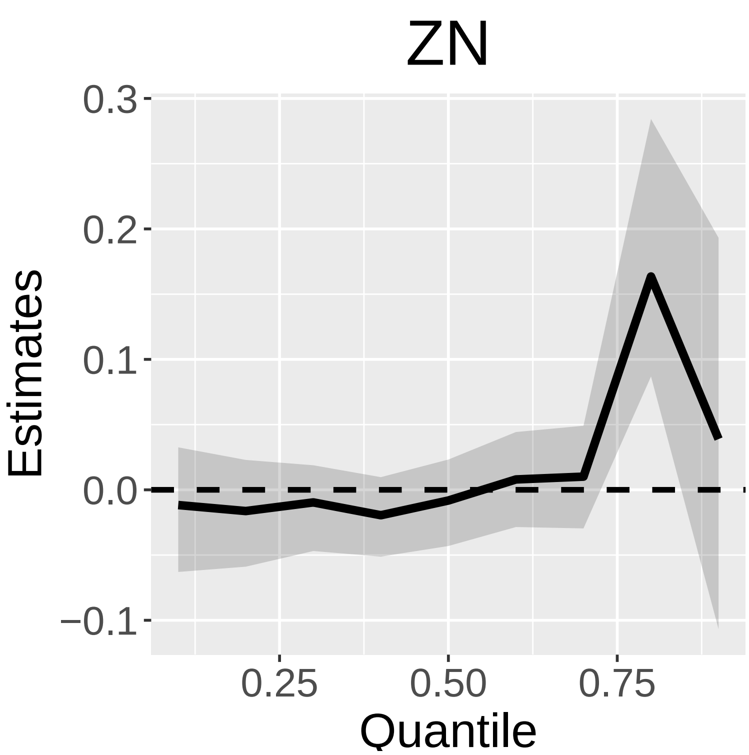
```r
plot_qr( res, 1 )
```



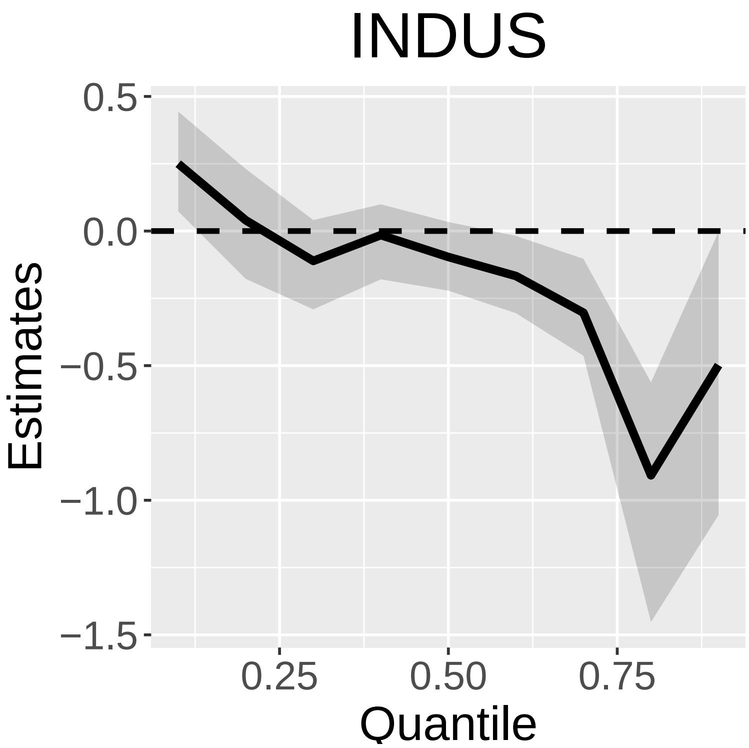
```
plot_qr( res, 2 )
```



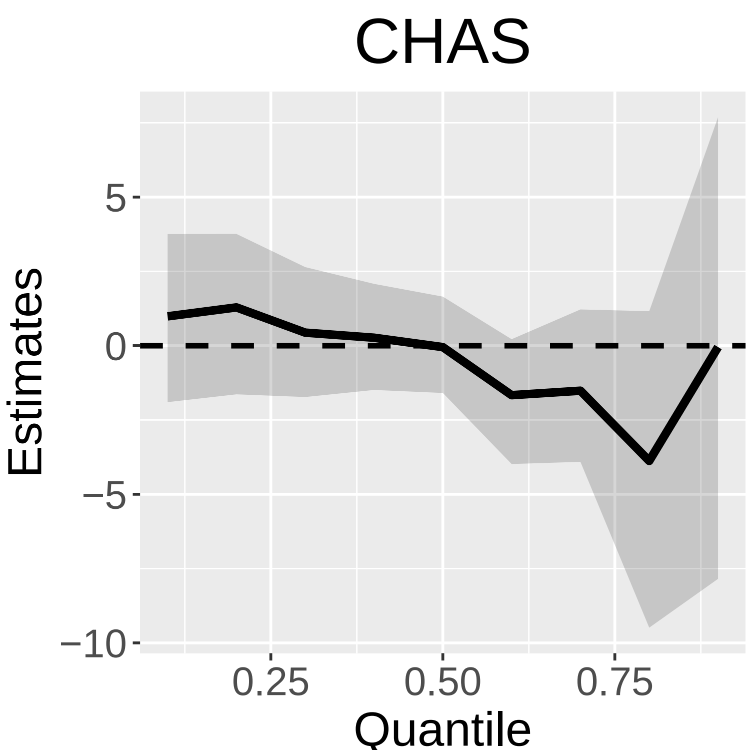

```
plot_qr( res, 3 )
```



```
plot_qr( res, 4 )
```

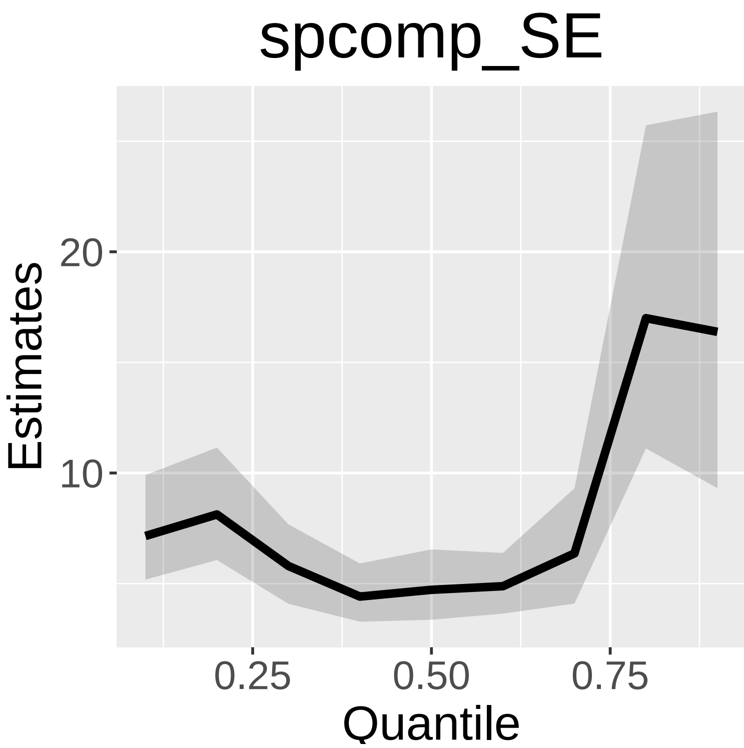



```
plot_qr( res, 5 )
```

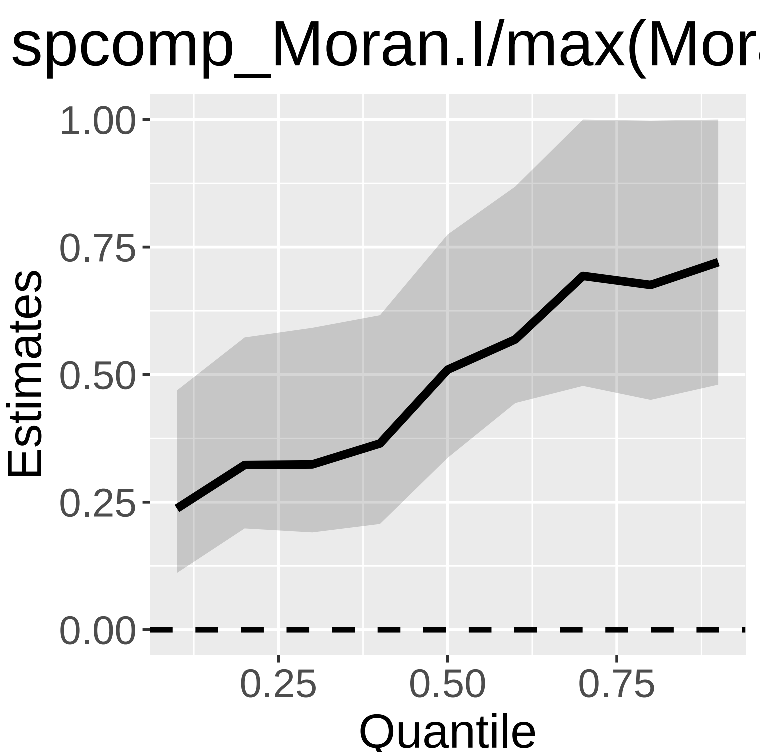


Standard errors and the scaled Moran coefficient (Moran.I/max(Moran.I)), which is a measure of spatial scale by quantile, are plotted if par = "s" is added. Here are the plots:

```r
plot_qr( res, par = "s" , 1 )
```




```r
plot_qr( res, par = "s" , 2 )
```



5 Low rank spatial econometric models

While ESF/RE-ESF and their extensions approximate Gaussian processes, this section explains low rank spatial econometric models approximating spatial econometric models (see Murakami et al., 2018).

5.1 Spatial weight matrix and their eigenvectors

The low rank models use eigenvectors and eigenvalues of a spatial connectivity matrix, which is called a spatial weight matrix or W matrix in spatial econometrics. The weigen function is available for the eigen-decomposition. Here is a code extracting the eigenvectors and eigenvalues from spatial polygons:

```
data( boston )
poly    <- readOGR( system.file( "shapes/boston_tracts.shp", package = "spData" )[ 1 ] )
```

```
## OGR data source with driver: ESRI Shapefile
## Source: "/Library/Frameworks/R.framework/Versions/4.0/Resources/library/spData/shapes/boston_tracts.s
## with 506 features
## It has 36 fields
```

```
weig    <- weigen( poly )            #### Rook adjacency-based W
```

By default, the weigen function returns a Rook adjacency-based W matrix. Other than that, knn-based W, Delaunay triangulation-based W, and user-specified W are also available.

5.2 Models

5.2.1 Low rank spatial lag model

The low rank spatial lag model (LSLM) approximates the following model:

$$y_i = \beta_0 + z_i + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma^2) z_i = \rho \sum_{i \neq j}^{N} w_{i,j} z_j + \sum_{k \neq 1}^{K} x_{i,k}\beta_k + u_i \qquad u_i \sim N(0, \tau^2)$$

where z_i is defined by the classical spatial lag model (SLM; see LeSage and Pace, 2009) with parameters ρ and τ^2. Just like the original SLM, ρ takes a value between 1 and $1/\lambda_N (< 0)$. Larger positive ρ means stronger positive dependence. τ^2 represents the variance of the SLM-based spatial process (i.e., z_i), while σ^2 represents the variance of the data noise ϵ_i. Because of the additional noise term, the LSLM estimates are different from the original SLM, in particular if data is noisy.

The LSLM is implemented using the lslm function. Here is a sample code:

```
y       <- boston.c[, "CMEDV" ]
x       <- boston.c[,c("CRIM","ZN","INDUS", "CHAS", "NOX","RM", "AGE")]
coords<- boston.c[,c("LON","LAT")]
res    <- lslm( y = y, x = x, weig = weig, boot = TRUE )
```

```
## [1] "------- Complete:20/200 -------"
## [1] "------- Complete:40/200 -------"
## [1] "------- Complete:60/200 -------"
## [1] "------- Complete:80/200 -------"
## [1] "------- Complete:100/200 -------"
## [1] "------- Complete:120/200 -------"
## [1] "------- Complete:140/200 -------"
## [1] "------- Complete:160/200 -------"
```

```
## [1] "------- Complete:180/200 -------"
## [1] "------- Complete:200/200 -------"
```

If boot=TRUE, a nonparametric bootstrapping is performed to estimate the 95% confidence intervals for the τ^2 and ρ parameters and the direct and indirect effects, which quantify spill-over effects. Default is FALSE. Here is the output in which {s_rho, sp_SE} are parameters $\{\rho, \tau^2\}$:

```
res
```

```
## Call:
## lslm(y = y, x = x, weig = weig, boot = TRUE)
##
## ----Coefficients------------------------------
##                 Estimate         SE    t_value      p_value
## (Intercept) -14.719039676 2.82212543 -5.2155866 2.748705e-07
## CRIM         -0.107615211 0.02851293 -3.7742599 1.809488e-04
## ZN            0.002594642 0.01276738  0.2032243 8.390474e-01
## INDUS        -0.098604511 0.06191541 -1.5925681 1.119273e-01
## CHAS          1.903178819 0.89128954  2.1353093 3.325050e-02
## NOX          -5.101316236 3.84673642 -1.3261414 1.854349e-01
## RM            6.922743307 0.33388005 20.7342228 0.000000e+00
## AGE          -0.040691404 0.01262483 -3.2231248 1.355874e-03
##
## ----Spatial effects (lag)---------------------
##         Estimates   CI_lower   CI_upper
## sp_rho 0.02709059 -0.0146306 0.06430701
## sp_SE  7.54450065  6.4809318 8.51169605
##
## ----Effects estimates-------------------------
##
## Direct:
##          Estimates     CI_lower    CI_upper p_value
## CRIM  -0.107999852  -0.16441926 -0.05561293    0.00
## ZN     0.002603915  -0.01924372  0.02758364    0.81
## INDUS -0.098956945  -0.20824364  0.02298966    0.13
## CHAS   1.909981199   0.15807234  3.62478157    0.03
## NOX   -5.119549463 -11.66897693  2.21719442    0.14
## RM     6.947486715   6.30060570  7.48111099    0.00
## AGE   -0.040836844  -0.06602344 -0.01599745    0.00
##
## Indirect:
##           Estimates      CI_lower     CI_upper p_value
## CRIM  -2.227815e-03 -0.0061712361 0.0012565506    0.21
## ZN     5.371341e-05 -0.0005488453 0.0007323576    0.80
## INDUS -2.041278e-03 -0.0066341510 0.0012106183    0.32
## CHAS   3.939898e-02 -0.0241310716 0.1195924986    0.24
## NOX   -1.056058e-01 -0.4204080182 0.0844118839    0.33
## RM     1.433123e-01 -0.0754791448 0.3284478587    0.21
## AGE   -8.423800e-04 -0.0023769406 0.0004806784    0.21
##
## ----Error statistics--------------------------
##                     stat
## resid_SE       3.9555161
## adjR2(cond)    0.8129243
## rlogLik     -1561.3219098
## AIC          3144.6438195
```

```
## BIC           3191.1357229
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

5.2.2 Low rank spatial error model

The low rank spatial error model (LSEM) approximates the following model:

$$y_i = \beta_0 + z_i + \epsilon_i \quad \epsilon_i \sim N(0, \sigma^2) z_i = \sum_{k \neq 1}^{K} x_{i,k}\beta_k + e_i \quad e_i = \lambda \sum_{i \neq j}^{N} w_{i,j} e_j + u_i \quad u_i \sim N(0, \tau^2)$$

where z_i is defined by the classical spatial error model (SLM) with parameters λ and τ^2. Just like the original SEM, λ takes a larger positive value in the presence of stronger positive dependence. τ^2 represents the variance of the SEM-based spatial process (i.e., z_i). As with LSLM, the LSEM estimates can be different from the original SEM if data is noisy.

The lsem function estimates LSEM, as follows:

```
data(boston)
res   <- lsem( y = y, x = x, weig = weig )
res
```

```
## Call:
## lsem(y = y, x = x, weig = weig)
##
## ----Coefficients------------------------------
##                 Estimate         SE    t_value      p_value
## (Intercept) -15.535928399 2.82054020 -5.5081393 6.082512e-08
## CRIM         -0.093112127 0.02911351 -3.1982447 1.479351e-03
## ZN            0.002300116 0.01292558  0.1779507 8.588411e-01
## INDUS        -0.063433279 0.06176206 -1.0270591 3.049394e-01
## CHAS          1.335521734 0.88216035  1.5139217 1.307414e-01
## NOX          -5.717186159 3.86329642 -1.4798725 1.396007e-01
## RM            7.052094665 0.33425292 21.0980796 0.000000e+00
## AGE          -0.037131943 0.01253448 -2.9623833 3.212894e-03
##
## ----Spatial effects (residuals)----------------
##           Estimates
## sp_lambda  0.885701
## sp_SE      2.926975
##
## ----Error statistics--------------------------
##                    stat
## resid_SE      4.0001174
## adjR2(cond)   0.8086816
## rlogLik   -1544.3307054
## AIC        3110.6614108
## BIC        3157.1533142
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

{s_lambda, sp_SE} are parameters $\{\lambda, \tau^2\}$.

6 Tips for modeling large samples

6.1 Eigen-decomposition

The meigen function implements an eigen-decomposition that is slow for large samples. For fast eigen-approximation, the meigen_f function is available. By default, this function approximates 200 eigenvectors; 200 is based on simulation results in Murakami and Griffith (2019a). The computation is further accelerated by reducing the number of eigenvectors. It is achieved by specifying enum by a number smaller than 200. While the meigen function took 243.8 seconds for 5,000 samples, the meigen_f took less than 1 second, as demonstrated below:

```
coords_test      <- cbind( rnorm( 5000 ), rnorm( 5000 ) )
system.time( meig_test200   <- meigen_f( coords = coords_test ))[3]
```

```
## elapsed
##   0.315
```

```
system.time( meig_test100   <- meigen_f( coords = coords_test, enum=100 ))[3]
```

```
## elapsed
##   0.109
```

```
system.time( meig_test50    <- meigen_f( coords = coords_test, enum=50 ))[3]
```

```
## elapsed
##   0.057
```

On the other hand, the weigen function implements the ARPACK routine for fast eigen-decomposition by default. The computational times with 5,000 samples and enum = 200 (default), 100, and 50 are as follows:

```
system.time( weig_test200   <- weigen( coords_test ))[3]
```

```
## elapsed
##   7.294
```

```
system.time( weig_test100   <- weigen( coords_test, enum=100 ))[3]
```

```
## elapsed
##   2.496
```

```
system.time( weig_test50    <- weigen( coords_test, enum=50 ))[3]
```

```
## elapsed
##   0.858
```

6.2 Parameter estimation

The basic ESF model is estimated computationally efficiently by specifying fn = "all" in the esf function. This setting is acceptable for large samples (Murakami and Griffith, 2019a). The resf and resf_vc functions estimate all the models explained above using a fast estimation algorithm developed in Murakami and Griffith (2019b). They are available for large samples (e.g., 100,000 samples). Although the SF-UQR model requires a bootstrapping to estimate confidential intervals for the coefficients, the computational cost for the iteration does not depend on sample size. Therefore, it is available for large samples too.

6.3 For very large samples (e.g., millions of samples)

A computational limitation is the memory consumption of the meigen and meigen_f functions to store Moran eigenvectors. Because of the limitation, the resf and resf_vc functions are not available for very large samples (e.g., millions of samples). To overcome this limitation, the besf and besf_vc functions perform the same calculation as resf and resf_vc but without saving the eigenvectors in the memory. Besides, for fast computation, these functions perform a parallel model estimation (see Murakami and Griffith, 2019c).

Here is an example implementing a spatial regression model using the besf function and an SVC model using the besf_vc function:

```
data(house)
dat   <- data.frame(coordinates(house),
                   house@data[,c("price","age","rooms","beds","syear")])
coords<- dat[ ,c("long","lat")]
y       <- log(dat[,"price"])
x     <- dat[,c("age","rooms","beds","syear")]
res1    <- besf(y=y, x=x, coords=coords)
res1
```

```
## Call:
## besf(y = y, x = x, coords = coords)
##
## ----Coefficients------------------------------
##                Estimate          SE    t_value       p_value
## (Intercept) -59.37567668 2.581150582 -23.003569 4.293192e-117
## age          -0.74288786 0.013258578 -56.030733  0.000000e+00
## rooms         0.10990599 0.002947034  37.293765 2.071132e-304
## beds          0.04868144 0.005001716   9.732947  2.181789e-22
## syear         0.03506390 0.001293217  27.113710 6.786691e-162
##
## ----Variance parameter------------------------
##
## Spatial effects (residuals):
##                     (Intercept)
## random_SE            0.05100897
## Moran.I/max(Moran.I)  0.37662789
##
## ----Error statistics--------------------------
##                     stat
## resid_SE       0.3365628
## adjR2(cond)    0.8053570
## rlogLik     -8908.3460444
## AIC         17832.6920887
## BIC         17897.8185696
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x)
```

```
res2    <- besf_vc(y=y, x=x, coords=coords)
```

```
## [1] "-------  Iteration 1  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
```

```
## [1] "5/5"
## [1] "BIC: 16489.568"
## [1] "-------  Iteration 2  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 16072.331"
## [1] "-------  Iteration 3  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 16071.434"
## [1] "-------  Iteration 4  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 16071.429"
## [1] "-------  Iteration 5  -------"
## [1] "1/5"
## [1] "2/5"
## [1] "3/5"
## [1] "4/5"
## [1] "5/5"
## [1] "BIC: 16071.429"
```

```
res2
```

```
## Call:
## besf_vc(y = y, x = x, coords = coords)
##
## ----Spatially varying coefficients on x (summary)----
##
## Coefficient estimates:
##   (Intercept)          age               rooms               beds
##  Min.   :-60.98   Min.   :-3.1941   Min.   :0.02172   Min.   :0.04803
##  1st Qu.:-60.18   1st Qu.:-1.0115   1st Qu.:0.07952   1st Qu.:0.04803
##  Median :-59.99   Median :-0.6916   Median :0.09550   Median :0.04803
##  Mean   :-60.00   Mean   :-0.7361   Mean   :0.09838   Mean   :0.04803
##  3rd Qu.:-59.78   3rd Qu.:-0.4152   3rd Qu.:0.11108   3rd Qu.:0.04803
##  Max.   :-59.26   Max.   : 1.5767   Max.   :0.27046   Max.   :0.04803
##      syear
##  Min.   :0.03543
##  1st Qu.:0.03543
##  Median :0.03543
##  Mean   :0.03543
##  3rd Qu.:0.03543
##  Max.   :0.03543
##
## Statistical significance:
```

```
##                          Intercept   age rooms  beds syear
## Not significant                  0  3742    76     0     0
## Significant (10% level)          0   994    99     0     0
## Significant ( 5% level)          0  1893   469     0     0
## Significant ( 1% level)      25357 18728 24713 25357 25357
##
## ----Variance parameters----------------------------------
##
## Spatial effects (coefficients on x):
##                       (Intercept)        age       rooms beds syear
## random_SE              0.04168388 0.07501517 0.005003324    0     0
## Moran.I/max(Moran.I)  0.19249460 0.10259899 0.058128227   NA    NA
##
## ----Error statistics-------------------------------------
##                       stat
## resid_SE        0.3188346
## adjR2(cond)     0.8252946
## rlogLik     -7974.8697582
## AIC         15973.7395165
## BIC         16071.4292377
##
## Note: The AIC and BIC values are based on the restricted likelihood.
##       Use method ="ml" for comparison of models with different fixed effects (x and xconst)
```

Roughly speaking, these functions are faster than the resf and resf_vc functions if the sample size is more than 100,000.

I have evaluated the computational time for an SVC modeling using the besf_vc function using a Mac Pro (3.5 GHz, 12-Core Intel Xeon E5 processor with 64 GB memory). The besf_vc function took only 8.0 minutes to estimate the 7 SVCs from 1 million samples. I also confirmed that besf_vc took 70.3 minutes to estimate the same model from 10 million samples. besf and besf_vc are suitable for very large data analysis.

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